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THE TAKENS–BOGDANOV BIFURCATION WITH O(2)-SYMMETRY

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The versal deformation of a vector field of co-dimension two that is equivariant under a representation of the symmetry group O(2) and has a nilpotent linearization at the origin is studied. An appropriate scaling allows us to formulate the problem in terms of a central-force problem with a small dissipative perturbation. We derive and analyse averaged equations for the angular momentum and the energy of the classical motion. The unfolded system possesses four different types of non-trivial solutions: a steady-state and three others, which are referred to in a wave context as travelling waves, standing waves and modulated waves. The plane of unfolding parameters is divided into a number of regions by (approximately) straight lines corresponding to primary and secondary bifurcations. Crossing one of these lines leads to the appearance or disappearance of a particular solution. We locate secondary saddle-node, Hopf and pitchfork bifurcations as well as three different global, i.e. homoclinic and heteroclinic, bifurcations.

1. Introduction

As has long been recognized, multiple (or degenerate) bifurcations hold the key to the understanding of the origin of complicated behaviour in physical systems (see, for example, Guckenheimer 1984). Near such bifurcations, secondary and higher-order bifurcations leading to such behaviour are accessible to a large extent analytically, and a complete analysis of the system is often possible. This is because near such bifurcations the system is described by a low-dimensional set of ordinary differential equations, resulting in a dramatic reduction of the number of degrees of freedom. This is particularly valuable for studies of systems that are described by partial differential equations (i.e. that have an infinite number of degrees of freedom). Typically, it has been found that the phenomena predicted by such a local study are robust in the sense that they persist for parameter values substantially far from those required for the degeneracy. Thus, even though multiple bifurcations occur at special parameter values, the analysis of their unfolding enables one to enumerate the variety of possible behaviour of a system. For steady-state bifurcations, this programme can be carried out completely by locating the 'organizing centre' for all the bifurcation phenomena, and unfolding this singularity using the techniques of singularity theory (Golubitsky & Schaeffer 1984). The theory is less complete in the case of bifurcations to dynamical behaviour, partly because a number of issues concerning structural stability and genericity remain unsolved. None the less, those techniques have met with a number of successes, including the analyses of the buckling of plates (Schaeffer & Golubitsky 1979), the stirred-tank chemical reactor (Golubitsky & Keyfitz 1980; Dangelmayr & Stewart 1986), optically bistable and tristable systems (Armbruster 1983; Armbruster & Dangelmayr 1985) and lasers with a saturable absorber (Dangelmayr *et al.* 1985), and understanding the bifurcation phenomena in particle physics, solidification and melting (Geiger *et al.* 1985) and in the dynamics and pattern selection in a number of convection systems (Knobloch & Proctor 1981; Knobloch & Guckenheimer 1983; Golubitsky *et al.* 1984; Arneodo *et al.* 1985), as well as the variety of states and the transitions between them in the Taylor–Couette system (Golubitsky & Stewart 1986; Chossat & Iooss 1985).

Recently it has been realized that systems possessing a symmetry undergo generically multiple bifurcations (Sattinger 1979). Such systems are therefore expected to exhibit more complicated behaviour. However, the presence of the symmetry facilitates the analysis of such bifurcations by restricting the structure of the 'normal form' equations, and therefore makes the study of equivariant systems particularly rewarding.

Several symmetries occur with great frequency in physical systems. The commonest is the

Z(2) reflectional symmetry that arises for example in the plate buckling problem, or in convective systems in which there is no distinction between clockwise and counterclockwise motion. Nearly as common is the symmetry O(2), the symmetry of the circle under rotations and reflections. Apart from systems with an obvious O(2)-symmetry, such as the motion of a suspended flexible tube with water running through it (Bajaj 1982), or the motion of water in a circular dish oscillated vertically (Ciliberto & Gollub 1985), O(2)-symmetry also arises naturally in all spatially periodic systems. Consider, for example, a continuous system on a line, with no distinction between left and right, and seek spatially periodic solutions. If we write

$$\psi(x) = w_k \, e^{ikx} + \text{c.c.}, \tag{1.1}$$

where k is the wave number of the unstable mode, then invariance with respect to translations by an amount d: $x \to x+d$ induces the action

$$\text{SO}(2): w_k \to e^{ikd} w_k, \tag{1.2}$$

i.e. rotation, whereas invariance with respect to reflection in $x = 0$ induces the action

$$\text{Z}(2): w_k \to \overline{w}_k. \tag{1.3}$$

Hence the dynamical system for the state vector w_k must be equivariant with respect to the group $\text{SO}(2) \times \text{Z}(2) \approx \text{O}(2)$.

In this paper we study a bifurcation with O(2)-symmetry that has a nilpotent linearization at the origin. This bifurcation is a generalization to O(2)-equivariant systems of the Takens–Bogdanov bifurcation with the linearization (in Jordan normal form) given by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{1.4}$$

In the unfolding of the Takens–Bogdanov bifurcations both Hopf and steady-state bifurcations occur, and the existence of a homoclinic connection can be established (Takens 1974). Both homoclinic and heteroclinic connections can be established in the Z(2)-equivariant problem (Knobloch & Proctor 1981). The O(2)-equivariant problem is even more interesting because the variety of secondary and global bifurcations is substantially larger. Because of the O(2)-symmetry, the Hopf bifurcation gives rise to what would be travelling and standing waves in the continuous system (1.1). Both situations break the O(2)-symmetry, the travelling waves being invariant under SO(2) and the standing waves under Z(2). In this paper, we adopt the wave terminology for the solutions with these symmetries. In addition, we locate secondary bifurcations that give rise to two-frequency waves (modulated waves) and study their interactions with the steady states produced at a simple bifurcation.

The Takens–Bogdanov bifurcation with equivariance under O(2)-symmetry is also considered by Guckenheimer (1986); however, only the solutions which are invariant under Z(2) are discussed, i.e. the standing waves and the steady state. The analysis of these solutions is accomplished by the fact that in the fixed point set of the reflection contained in O(2) the problem reduces to a standard Takens–Bogdanov bifurcation with Z(2)-symmetry. In this simplification it is, however, not possible to deduce the stabilities of the two Z(2)-invariant solutions with respect to variations transverse to the fixed point set. We are able to give a complete description of all the different solutions, including the travelling waves and the modulated waves, and their stability properties.

The plan of the paper is as follows. In §2, we formulate the O(2)-equivariant problem and derive the normal form equation. In §3, a scaling of the variables is used to obtain a truncated normal form, which is then analysed in detail. The analysis is based on the observation that with the scaling used the normal form is approximately hamiltonian. The hamiltonian system can be solved completely by means of elliptic functions, and persistence of closed orbits under the non-hamiltonian 'perturbation' is used to identify the solutions of the truncated normal form. The development of this theory is presented in §§4–6 and follows the methodology used by Knobloch & Proctor (1981) in their study of the Z(2)-equivariant problem. Throughout the paper we find it useful to exploit a formal similarity of the present problem with a classical dynamics problem: motion of a particle in a central force field, subject to small dissipation. This similarity is not accidental, and is forced on us by the O(2) symmetry. Our results are summarized in the form of bifurcation diagrams in the plane of the two unfolding parameters (see §7). A discussion of the results is presented in §8. A number of applications of the theory is immediate, and those are also mentioned in §8. Certain mathematical details are relegated to the Appendixes.

2. Formulation of the problem

We seek a vector field that is equivariant under a representation of the symmetry group O(2), the symmetry group of rotations and reflections of the circle, and has a nilpotent linearization at the origin leading to both Hopf and steady-state bifurcations upon unfolding. In general (see, for example, Golubitsky & Stewart 1985), there are two situations in which a vector field, equivariant under a representation of a group Γ, can possess Hopf bifurcations: either Γ acts via the diagonal action on $\mathbb{R}^m \oplus \mathbb{R}^m$ with an absolutely irreducible action on $\mathbb{R}^m$, or Γ acts irreducibly but not absolutely irreducibly on $\mathbb{R}^m$. In our case, the additional requirement of a non-zero but nilpotent linear part selects the former action, that is, we need the diagonal action of O(2) on $\mathbb{R}^2 \oplus \mathbb{R}^2$ which is absolutely irreducible on $\mathbb{R}^2$. The linearized part then has the form

$$L_0 = \begin{bmatrix} 0_2 & 1_2 \\ 0_2 & 0_2 \end{bmatrix}, \tag{2.1}$$

where 0_2 and 1_2 are, respectively, the zero and the identity operators on $\mathbb{R}^2$. This is the appropriate generalization of (1.4). In complex coordinates, the desired action of O(2) is represented by the operations

$$\text{rotation } (v, w) \to (e^{i\theta} v, e^{i\theta} w), \tag{2.2a}$$

$$\text{reflection } (v, w) \to (\bar{v}, \bar{w}), \tag{2.2b}$$

acting on complex vectors $(v, w) \in \mathbb{C}^2 \approx \mathbb{R}^4$. There are three basic functions that are invariant under (2.2):

$$\sigma_1 = |v|^2, \quad \sigma_2 = |w|^2, \quad \sigma_3 = v\bar{w} + \bar{v}w. \tag{2.3}$$

Any other invariant function is of the form $f(\sigma_1, \sigma_2, \sigma_3)$ for some $f: \mathbb{R}^3 \to \mathbb{R}$. A general smooth vector field that is equivariant under the representation (1.2) is given by

$$\left.\begin{aligned} \dot{v} &= g_1 v + g_2 w, \\ \dot{w} &= g_3 v + g_4 w, \end{aligned}\right\} \tag{2.4}$$

where $g_j = g_j(\sigma_1, \sigma_2, \sigma_3)$, $j = 1, 2, 3, 4$ are smooth real functions on $\mathbb{R}^3$.

In this paper, we seek to study the dynamics of a versal unfolding of a system of the form (2.4) near the origin $(v, w) = (0, 0)$, which has the nilpotent linearization (2.1). We proceed by expanding the functions g_j $(j = 1, 2, 3, 4)$ in a Taylor series to second order in v, w, and neglecting higher-order terms. We obtain

$$\left.\begin{aligned} \dot{v} &= w + (a_1|v|^2 + b_1|w|^2)\, v + c_1 v^2\bar{w} + (a_2|v|^2 + b_2|w|^2)\, w + c_2 \bar{v}w^2, \\ \dot{w} &= (a_3|v|^2 + b_3|w|^2)\, v + c_3 v^2\bar{w} + (a_4|v|^2 + b_4|w|^2)\, w + c_4 \bar{v}w^2. \end{aligned}\right\} \tag{2.5}$$

Here a_i, b_i, c_i $(i = 1, 2, 3, 4)$ are real coefficients. It is convenient (indeed, essential) to simplify these equations by means of a near-identity O(2)-equivariant coordinate change of the form

$$\left.\begin{aligned} v &= v' + (\alpha_1|v|^2 + \beta_1|w|^2)\, v + \gamma_1 v^2\bar{w} + (\alpha_2|v|^2 + \beta_2|w|^2)\, w + \gamma_2 \bar{v}w^2, \\ w &= w' + (\alpha_3|v|^2 + \beta_3|w|^2)\, v + \gamma_3 v^2\bar{w} + (\alpha_4|v|^2 + \beta_4|w|^2)\, w + \gamma_4 \bar{v}w^2, \end{aligned}\right\} \tag{2.6}$$

where the real coefficients α_i, β_i, γ_i $(i = 1, 2, 3, 4)$ are to be chosen to achieve maximum simplification of (2.5). We first write (2.5) in the primed variables:

$$\begin{aligned} \dot{v}' = {} & w' + (a_1 + \alpha_3)|v'|^2 v' + (a_2 - 2\alpha_1 + \alpha_4)|v'|^2 w' \\ & + (b_1 - \alpha_2 + \beta_3 - 2\gamma_1)|w'|^2 v' + (b_2 - \beta_1 + \beta_4 - \gamma_2)|w'|^2 w' \\ & + (c_1 - \alpha_1 + \gamma_3)\, v'^2\bar{w}' + (c_2 - \alpha_2 + \gamma_4)\, w'^2\bar{v}' + O(5), \end{aligned} \tag{2.7a}$$

$$\begin{aligned} \dot{w}' = {} & a_3|v'|^2 v' + (a_4 - 2\alpha_3)|v'|^2 w' + (b_3 - \alpha_4 - 2\gamma_3)|w'|^2 v' \\ & + (b_4 - \beta_3 - \gamma_4)|w'|^2 w' + (c_3 - \alpha_3)\, v'^2\bar{w}' + (c_4 - \alpha_4)\, w'^2\bar{v}' \\ & + O(5). \end{aligned} \tag{2.7b}$$

We find it convenient to choose α_i, β_i, γ_i $(i = 1, 2, 3, 4)$ to eliminate all cubic terms in (2.7a). In particular, we choose $\alpha_3 = -a_1$. We also choose $\alpha_4 = c_4$. Because α_4 is chosen to eliminate the second term in (2.7a), we must take $\alpha_1 = \frac{1}{2}(a_2 + c_4)$, and hence to eliminate the fifth term in (2.7a) we require $\gamma_3 = -c_1 + \frac{1}{2}(a_2 + c_4)$. This implies that the first three terms as well as the fifth term in (2.7b) cannot be removed by the coordinate change (2.6). There is no obstruction to the elimination of the fourth term in (2.7b). To third order, we are therefore left with the dynamical system

$$\left.\begin{aligned} \dot{v} &= w, \\ \dot{w} &= [A|v|^2 + B|w|^2 + C(v\bar{w} + \bar{v}w)]\, v + D|v|^2 w, \end{aligned}\right\} \tag{2.8}$$

where

$$\left.\begin{aligned} A &= a_3, \quad B = b_3 - a_2 + 2c_1 - 2c_4, \\ C &= c_3 + a_1, \quad D = a_4 + a_1 - c_3. \end{aligned}\right\} \tag{2.9}$$

In writing (2.8) we have dropped the primes and all higher-order terms.

To study the bifurcations associated with the linearization (2.3), we need to unfold the system (2.8). We do this by adding small linear equivariant terms to (2.8). Without loss of generality we write

$$\left.\begin{aligned} \dot{v} &= w, \\ \dot{w} &= \mu v + \nu w + [A|v|^2 + B|w|^2 + C(v\bar{w} + \bar{v}w)]\, v + D|v|^2 w. \end{aligned}\right\} \tag{2.10}$$

We shall refer to $\mu, \nu \in \mathbb{R}$ as the unfolding parameters, and to (2.10) as the normal form for this O(2)-equivariant bifurcation problem.

Some remarks are in order concerning structural stability and genericity. We use a weak form of structural stability, namely that the qualitative structure of the bifurcation set in the space of unfolding parameters (μ, ν) be preserved under perturbation. In this paper, the bifurcation set is defined by the collection of those lines in the (μ, ν)-plane where bifurcations to steady states, standing and travelling waves and modulated waves take place. For the normal form (2.10), these lines turn out to be straight. Higher-order terms will produce curved lines but the slopes at the origin are preserved, provided a finite set of non-degeneracy conditions for the coefficients A, C, D are satisfied (see §7). This follows directly from the calculations carried out in the subsequent sections and justifies the neglect of higher-order terms. From that point of view the system (2.10) is also generic (up to the relevant co-dimension in O(2) systems) because two non-degeneracy conditions for the linear part and a set of non-degeneracy conditions for the cubic Taylor terms must be satisfied by an arbitrary $(m \geqslant 2)$-parameter family of vector fields. Hence, given, for example, a three-parameter family of O(2)-equivariant vector fields, we expect the system under discussion to occur on certain smooth lines in the space of parameters.

3. Elementary considerations

In this section, we begin our study of the dynamics described by the normal form (2.10). We first show that such a study can be undertaken analytically by a proper rescaling of (2.10), and then show that it can be cast into the form of a central-force problem with a slowly varying energy and angular momentum. We introduce the elementary solutions to this problem, the steady states and the travelling (rotating) waves, interpret them in terms of the motion of a particle in a potential, and describe their stability properties. Other solutions (standing and modulated waves), whose existence can be established by means of the method of averaging, are deferred to §4.

3.1. *Scaling of the normal form equation*

The appropriate scaling that renders the present problem analytically tractable is suggested by a previous study of the Z(2)-equivariant problem (Knobloch & Proctor 1981; Guckenheimer & Knobloch 1983). We introduce a slow time τ defined by

$$\tau = \epsilon t, \tag{3.1}$$

where ϵ is a small parameter, and scale v and the unfolding parameters according to

$$v \to \epsilon v, \quad \mu \to \epsilon^2 \mu, \quad \nu \to \epsilon^2 \nu. \tag{3.2}$$

With this scaling, the normal form (2.10) becomes

$$v'' - \epsilon[\nu v' + C(\bar{v}v' + v\bar{v}')\, v + D|v|^2 v'] - (\mu + A|v|^2)\, v = O(\epsilon^2). \tag{3.3}$$

Here the prime denotes differentiation with respect to τ. Note that the coefficient B does not enter at leading order in ϵ. The situations described below depend therefore only on the choice of the three coefficients A, C, D (and the unfolding parameters). This is in contrast with the Z(2)-equivariant problem, in which the solutions depend on two coefficients only.

We can cast (3.3) into a more familiar form by introducing real variables (r, ϕ) defined by

$$v = r\,e^{i\phi}; \tag{3.4}$$

(3.3) then becomes

$$r'' - r\phi'^2 - (\mu + Ar^2)\,r = \epsilon[\nu + (2C+D)\,r^2]\,r' + O(\epsilon^2), \tag{3.5a}$$

$$r\phi'' + 2r'\phi' = \epsilon(\nu + Dr^2)\,r\phi' + O(\epsilon^2). \tag{3.5b}$$

These equations suggest that we define the 'angular momentum'

$$L \equiv r^2\phi' \tag{3.6}$$

and the 'potential'

$$V \equiv \tfrac{1}{2}L^2/r^2 - \tfrac{1}{2}\mu r^2 - \tfrac{1}{4}Ar^4, \tag{3.7}$$

so that (3.5) can be written in the form

$$r'' + \partial V/\partial r = \epsilon(\nu + Mr^2)\,r' + O(\epsilon^2), \tag{3.8a}$$

$$L' = \epsilon(\nu + Dr^2)\,L + O(\epsilon^2), \tag{3.8b}$$

where $M = 2C + D$. Together with (3.6), the equation for the phase ϕ, the system (3.8) is equivalent to the complex equation (3.3). Note that when $\epsilon = 0$ this system describes the motion of a particle in a central force field with potential (3.7). This motion is specified by two constants of motion, the 'energy' E given by

$$E \equiv \tfrac{1}{2}r'^2 + V(r), \tag{3.9}$$

and the angular momentum L. In the limit $\epsilon \downarrow 0$, these two constants evolve on a superslow time scale $\epsilon\tau$, and this provides the basis for the method of averaging employed in §4.

3.2. *The potential $V(r)$*

The key to the understanding of the different types of solutions admitted by the system (3.8) is provided by the form of the potential (3.7). In this section, we sketch the potential for different choices of the parameters A, μ and L^2, and identify the solution types.

When $L^2 = 0$ there is no centrifugal force, and the motion is then purely in the radial direction. Typical potentials are shown in figure 1*a–c*. In figure 1*d, e* we show the effect of non-zero L^2. The resulting motion is an orbital motion in the ϕ-direction with a superposed oscillation in the radial direction.

These figures assist us in describing the following five simple solutions.

(i) *The trivial solution* (T), $r = 0$, $L^2 = 0$. This is the time-independent solution that exists for $E = 0$, and all values of the other parameters. Note that when $r = 0$, L^2 must also vanish. This solution is equivariant with respect to the full symmetry group O(2).

(ii) *The steady state* (SS), $r > 0$, $L^2 = 0$. This time-independent solution is present for energies corresponding to the extrema with $r \neq 0$ of the potential $V(r)$ for $L^2 = 0$ as in figure 1*a, c*. The steady state is taken into a whole circle of steady states by the action of O(2).

(iii) *The travelling waves* (TW) occur for $L^2 > 0$, and for energies corresponding to an extremum of the potential (cf. figure 1*d, e*). Hence, there is no oscillation in the radial direction, but because $L^2 > 0$, these solutions describe orbital motion with constant angular frequency L/r^2. We call these rotating (or travelling) waves. Because they can rotate in one or more directions they break the O(2)-symmetry, but respect the symmetry SO(2) $\subset$ O(2).

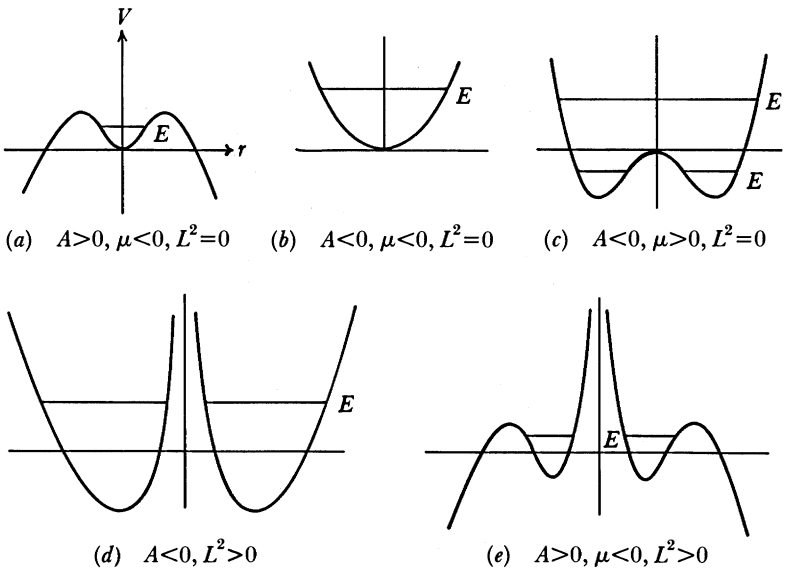

FIGURE 1. The potential $V(r)$ for different choices of the parameters.

(iv) *The standing wave* (SW) occurs for $L^2 = 0$, and is an oscillation in the radial direction. There are two types: oscillations about the trivial solution T as in figure 1*a*, *b*, and oscillations about a steady state (SS) as in figure 1*c*. Note that in the latter case both types of standing waves can occur, depending on the choice of the energy E. The standing waves also break the O(2)-symmetry but they respect the Z(2)-symmetry.

(v) *The modulated waves* (MW) occur for $L^2 > 0$ and energies that allow oscillations in the radial direction as well. The motion is a two-torus, with one frequency corresponding to the orbital motion, and the other to radial oscillations.

Note that both the standing waves and the modulated waves are parametrized by the energy E, and the latter by the angular momentum as well. Thus, further conditions are required to specify them uniquely. This is provided by the requirement that the small dissipative terms in (3.8) do not destroy the oscillation over time $\tau = o(\epsilon^{-1})$. This is accomplished by the method of averaging, and is deferred to §4.

3.3. *The elementary solutions and their stability*

We introduced above the three simplest solutions of the system (3.8). We now give a detailed description of each of them together with their stability properties.

(i) *Trivial state* (T)

This is the solution $r = 0$, $L^2 = 0$. Because of the O(2)-symmetry, it is present for all values of the parameters μ and ν. The stability of this solution is described by a quadratic eigenvalue equation for the growth rate λ

$$\lambda^2 - \epsilon\nu\lambda - \mu = 0. \tag{3.10}$$

A bifurcation producing steady-state solutions occurs when $\lambda = 0$, i.e. along the line

$$L_0 : \mu = 0. \tag{3.11}$$

A Hopf bifurcation occurs along the half-line

$$H_0: \nu = 0, \quad \mu < 0, \tag{3.12}$$

and leads to both travelling (rotating) waves, and standing waves. The O(2)-symmetry forces both types of solutions to appear simultaneously. The travelling waves break the Z(2)-symmetry of the trivial state, while the standing waves break the SO(2)-symmetry (cf. Golubitsky & Stewart 1985). As a function of μ and ν the two eigenvalues λ behave as follows: when $\mu > 0$, one is positive, and one negative; as μ decreases through zero, for $\nu > 0$ the negative eigenvalue becomes positive. For $\nu < 0$, μ decreasing through zero produces two negative eigenvalues. Finally, if $\mu < 0$, then for $\nu < 0$, there is a complex pair of eigenvalues with negative real part, which becomes positive as ν passes through zero.

(ii) *Steady state* (SS)

These O(2)-equivariant time-independent solutions have zero angular momentum, and a non-zero amplitude r_0 given by

$$\mu + Ar_0^2 = 0; \tag{3.13}$$

they bifurcate from the trivial solution at $\mu = 0$. Their stability is determined by linearizing the system (3.8) about the solution (3.13). We let

$$r = r_0(1+\eta), \quad L = \xi, \tag{3.14}$$

and obtain

$$\eta'' - \epsilon(\nu - (M/A)\mu)\,\eta' + 2\mu\eta = 0, \tag{3.15a}$$

$$\xi' = \epsilon(\nu - (D/A)\mu)\,\xi. \tag{3.15b}$$

The stability is described by the three eigenvalues of (3.15).

Apart from the bifurcation at $\mu = 0$ producing the SS-branch, the only bifurcation to time-independent solutions occurs when the eigenvalue λ_L in the L-direction vanishes. This bifurcation produces solutions characterized by constant energy and angular momentum, and we call such solutions travelling (rotating) waves. They bifurcate from the SS-branch along the half-line

$$L_m: A\nu = D\mu, \quad A\mu < 0. \tag{3.16}$$

When $A < 0$ there can also be a secondary Hopf bifurcation from the SS-branch occurring on the half-line

$$L_M: A\nu = M\mu, \quad \mu > 0 \quad \text{and} \quad A < 0. \tag{3.17}$$

This bifurcation produces small-amplitude oscillations about a non-zero steady state, and will be discussed in detail in §5. There are no other local bifurcations from the SS-branch.

To determine the stability assignments, we observe that when $A > 0$, the steady states exist only in $\mu < 0$. Then, of the two eigenvalues of (3.15a) one is positive and one negative. The L-eigenvalue is positive in $\{\nu > D\mu/A, \mu < 0\}$, and negative in $\{\nu < D\mu/A, \mu < 0\}$. When $A < 0$, the steady states lie in $\mu > 0$. Now, there are two eigenvalues with $\mathrm{Re}\,\lambda > 0$ in $\{\nu > M\mu/A, \mu > 0\}$ and $\mathrm{Re}\,\lambda < 0$ in $\{\nu < M\mu/A, \mu > 0\}$, whereas the third eigenvalue $\lambda_L > 0$ in $\{\nu > D\mu/A, \mu > 0\}$ and $\lambda_L < 0$ in $\{\nu < D\mu/A, \mu > 0\}$.

(iii) *Travelling (rotating) waves* (TW)

These solutions have a constant amplitude, r_0, and a constant circular frequency, $\phi' = \omega_0$, given by

$$\nu + Dr_0^2 = 0, \quad \omega_0^2 = -(\mu + Ar_0^2), \tag{3.18}$$

and exist provided r_0, ω_0 are both real. They bifurcate from the trivial solution at $\nu = 0$ with a branch of standing waves (see §5). We note that in the case $A > 0$, r_0 is a minimum of the potential $V(r)$ for $\mu < 3A\nu/2D$ and a maximum for $\mu > 3A\nu/2D$. In the case $A < 0$, r_0 is always a minimum of V (see figure 1*d*, *e*).

To study the stability properties of the travelling waves we let

$$r = r_0(1+\eta), \quad L = L_0(1+\xi), \tag{3.19}$$

where $L_0 = r_0^2\omega_0$, with r_0 and ω_0 given by (3.18). Linearizing the system (3.8) in η, ξ and eliminating ξ, we obtain the third-order equation

$$\eta''' + 2\epsilon\nu\frac{C}{D}\eta'' - (4\mu - (6A/D)\,\nu)\,\eta' - 4\epsilon\nu(\mu - (A/D)\,\nu) = 0. \tag{3.20}$$

There are steady-state bifurcations when $\nu = 0$ (the beginning of the TW-branch), and along the half-line $A\nu = D\mu$, $A\mu < 0$. When approaching this line, the circular frequency ω_0 of the travelling waves tends toward zero, and the TW-branch joins the SS-branch, as already explained in the preceding subsection.

It is also possible to have a secondary Hopf bifurcation from the TW-branch. To see this, let $\eta \propto \exp(i\omega\tau)$, and set the real and imaginary parts of (3.20) equal to zero. The resulting conditions may be written in the form

$$\omega^2 = -(4\mu - (6A/D)\,\nu) = (4D/(M-D))\,\omega_0^2. \tag{3.21}$$

Because $\omega_0^2 > 0$, the secondary Hopf bifurcation can only occur when $M/D > 1$. In the unfolding plane, this bifurcation lies along the half-line

$$L_H\colon \mu = [(3M-5D)/(2M-4D)]\frac{A}{D}\nu, \quad \mu < \frac{A}{D}\nu. \tag{3.22}$$

This Hopf bifurcation introduces an explicit time dependence into the dynamics: oscillations with the new frequency ω in both the amplitude r and the angular momentum L, and corresponds to the formation of a two-torus in the dynamics of (3.5). We shall call this new solution branch the branch of modulated waves (MW). Its properties are discussed in §6.

We now describe the stability assignments resulting from a linear stability analysis on the TW-branch. Because the branch bifurcates at $\nu = 0$, we expand the three eigenvalues λ of (3.20) in powers of ν:

$$\mathrm{Re}\,\lambda = \tfrac{1}{2}\nu\epsilon[(2D-M)/D] + O(\nu^2), \quad \lambda_3 = -\epsilon\nu + O(\nu^2). \tag{3.23}$$

For $D > 0$, the TW-branch lies between the half-lines H_0 and L_m in $\nu < 0$. We consider first the case $A > 0$. If $M < 2D$, there is no secondary Hopf bifurcation, and from (3.23) the stability assignments are $(--+)$. When $0 < 2D < M$, there is a secondary Hopf bifurcation, and the stability is $(+++)$ between H_0 and L_H, and $(--+)$ between L_H and L_m. When $D < 0$,

the branch lies in $\nu > 0$. Again, when $M > 2D$, there is no secondary bifurcation and the stability is $(++-)$; when $M < 2D < 0$, the assignments are $(---)$ between H_0 and L_H, and $(++-)$ between L_H and L_m. When $A < 0$, additional possibilities arise. Although there is still no secondary bifurcation when $M < D$ $(D > 0)$, it does occur when $0 < D < M < 2D$, and stability changes from $(--+)$ to $(+++)$. There is, however, no secondary bifurcation when $M > 2D > 0$, and the stability remains $(+++)$. When $D < 0$, there is no secondary bifurcation if $D < M$, but there is one when $0 > D > M > 2D$, and the stability changes from $(++-)$ to $(---)$. Finally, when $0 > 2D > M$, the TW-branch is stable throughout. These results will be used to draw the bifurcation diagrams (see §7).

This concludes our discussion of the elementary solutions. In the next section we turn our attention to the remaining two classes of solutions: the standing and modulated waves.

4. The averaged equations

For $\epsilon = 0$, the system (3.8) is an integrable hamiltonian system with E, L being the constants of the motion. Consequently, there is a two-parameter family of invariant tori in the four-dimensional phase space. If we ignore the decoupled phase, ϕ, the family of tori reduces to a two-parameter family of closed orbits in (r, r', L)-space. As explained in §3, we distinguish between closed orbits with $L = 0$ (SW) and orbits with $L \neq 0$ (MW). For $\epsilon \neq 0$, most of the closed orbits are destroyed, but some of them may persist under the non-hamiltonian perturbation. To find out which orbits persist, we apply the method of averaging. In this section we derive averaged equations for the classical constants of motion, E and L, by means of intuitive arguments. A formal justification is given in Appendix A. The analysis of the SW-solutions and MW-solutions is deferred to §§5 and 6, respectively.

4.1. *The structure of the* (E, L^2)*-plane*

Much of what follows is based on the functional dependence of r' on E, L^2 and μ in the integrable system (3.9). We introduce the variable $s = r^2$, and write

$$s' = 2\sqrt{[P(s)]}, \quad P(s) = \tfrac{1}{2}As^3 + \mu s^2 + 2Es - L^2. \tag{4.1}$$

Of particular importance is the root structure of the cubic $P(s)$. When P has three distinct real roots we denote them by s_i, $i = 1, 2, 3$, and order them according to $s_3 < s_2 < s_1$. Positive roots correspond to turning points of the classical motion. Two coalescing positive roots give rise to a stationary point of (3.9). If $L^2 = 0$ there is always one zero root, and two roots coalesce when $P(s) = \mathrm{d}P(s)/\mathrm{d}s = 0$. If confined to $s \geqslant 0$, the latter equations define a bifurcation set in the $(L^2 \geqslant 0, E, \mu)$-half space such that across it the number of positive roots changes by two. In figure 2 we have sketched the bifurcation set for fixed μ in the (L^2, E)-plane and the form of $P(s)$ in $s \geqslant 0$ for $L^2 > 0$ and $L^2 = 0$.

4.2. *Derivation of the averaged equations*

The non-conservative perturbations in (3.8) induce a slow rate of change of the energy and the angular momentum

$$E' = \epsilon(Mr^2 + \nu)\, r'^2 + \epsilon(L^2/r^2)(Dr^2 + \nu) + O(\epsilon^2), \tag{4.2}$$

$$L' = \epsilon(Dr^2 + \nu)\, L + O(\epsilon^2). \tag{3.8b}$$

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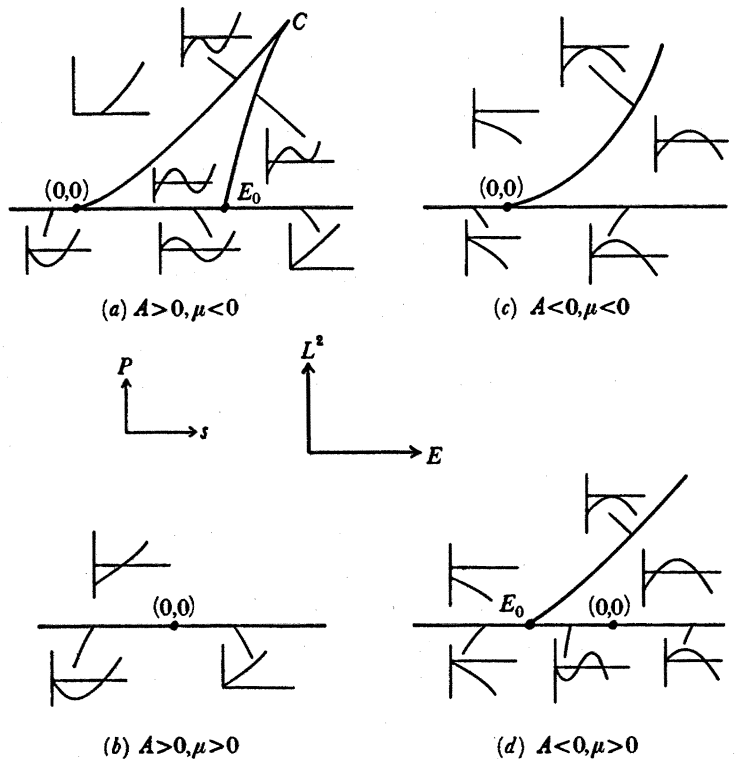

FIGURE 2. The bifurcation set in the (E, L^2)-plane for fixed μ and the root structure of $P(s)$ in $s \geqslant 0$. The bifurcation set is given by $(27A^2L^2-8\mu^3+36AE\mu)^2 = 64(\mu^2-3AE)^3$ with the restriction $E \geqslant 0$ if $\mu < 0$ and $\mu < 0$ if $A > 0$. In (a) and (d), $E_0 = \mu^2/4A$. The cusp point C in (a) is given by $(E, L) = (\mu^2/3A, -4\mu^3/27A^2)$.

In the method of averaging, the right-hand sides of (4.2) and (3.8b) are averaged over one period of a closed orbit corresponding to the unperturbed system, i.e. the radial variable is integrated out leaving a decoupled system for (E, L). Substituting $s' = 2\sqrt{[P(s)]}$ into the right-hand side of (4.2), (3.8b) and carrying out the s-integration yields the system

$$E' = \epsilon f(E, L^2), \tag{4.3a}$$

$$L' = \epsilon L g(E, L^2), \tag{4.3b}$$

where

$$f(E, L^2) = \frac{1}{J_0}\int_{s_-}^{s_+} \frac{\mathrm{d}s}{s}(Ms+\nu)\sqrt{[P(s)]} + L^2(D+\nu J_{-1}/J_0), \tag{4.4a}$$

$$g(E, L^2) = \nu + DJ_1/J_0. \tag{4.4b}$$

Here, the integration limits are the zeros of $P(s)$,

$$(s_-, s_+) = \begin{cases} (s_3, s_2) & \text{for} \quad A > 0 \\ (s_2, s_1) & \text{for} \quad A < 0 \end{cases}$$

(see figure 2), and the J_n are standard elliptic integrals, defined by

$$J_n = \int_{s_-}^{s_+} \mathrm{d}s \cdot s^n/\sqrt{[P(s)]}, \quad n = 0, 1, -1. \tag{4.5}$$

Note the symmetry $L \to -L$ of the system (4.3). The period of an underlying closed orbit corresponding to given (E, L) for $\epsilon = 0$ is J_0. There are recursion relations (see, for example, Gröbner & Hofreiter 1975) that allow one to reduce any integral containing rational functions and the square root of a cubic to a linear combination of the three integrals (4.5). Applying these relations to the first integral in (4.4*a*), we obtain

$$f(E, L^2) = \frac{4}{3}\left(\nu - \frac{M\mu}{5A}\right)E + (D - \tfrac{3}{5}M)\,L^2 + \left(\tfrac{1}{3}\mu\nu + \tfrac{4}{5}ME - \frac{4M}{15A}\mu^2\right)\frac{J_1}{J_0}. \tag{4.6}$$

Note that the J_{-1}-integral (an elliptic integral of the third kind), which appeared in (4.4*a*) has disappeared in (4.6). Equations (4.3) and (4.4*b*) are the averaged equations on which the analyses in §§5 and 6 are based. A more formal derivation of these equations in terms of action-angle variables is presented in Appendix A. There are theorems (see, for example, Guckenheimer & Holmes 1983) that state that hyperbolic fixed points and limit cycles in the averaged equations correspond to hyperbolic limit cycles and two tori in the original system of the same stability types. Hence, the averaged equations describe the essential features of the original system.

4.3. *The variables k and ρ*

Closed orbits of the original system (3.8) correspond to steady states of (4.3), i.e. to any (E, L) satisfying $f(E, L^2) = 0 = Lg(E, L^2)$ there corresponds a unique periodic orbit of the perturbed system (3.8) (a two torus for (3.5)), parametrized by ϵ, that tends for $\epsilon \downarrow 0$ to the periodic solution of the unperturbed system associated to the given values of (E, L). The stability assignment of such a closed orbit is the same as that of the corresponding stationary point of (4.3). Consequently, we have to analyse the solutions of $f = 0 = Lg$ to discuss the periodic solutions of (3.8). The SW-solutions satisfy $L = 0$, $f(E, 0) = 0$ and the MW-solutions are determined by $f(E, L^2) = 0 = g(E, L^2)$. For the analysis of these persistence conditions, it is convenient to express the standard elliptic integrals J_0, J_1 in terms of Legendre's canonical complete integrals $\mathscr{K}(k)$ and $\mathscr{E}(k)$ of the first and second kind, respectively, with modulus

$$k^2 = \begin{cases} (s_2 - s_3)/(s_1 - s_3) & \text{for} \quad A > 0; \\ (s_1 - s_2)/(s_1 - s_3) & \text{for} \quad A < 0; \end{cases} \qquad 0 \leqslant k \leqslant 1. \tag{4.7}$$

From Gröbner & Hofreiter (1975) we obtain

$$\begin{aligned} CJ_0(E, L^2, \mu) &= \mathscr{K}(k), \\ CJ_1(E, L^2, \mu) &= \begin{cases} s_1\mathscr{K}(k) - (s_1 - s_3)\,\mathscr{E}(k) & \text{for} \quad A > 0, \\ s_3\mathscr{K}(k) + (s_1 - s_3)\,\mathscr{E}(k) & \text{for} \quad A < 0, \end{cases} \end{aligned} \tag{4.8}$$

where

$$C = 2^{-\frac{3}{2}}[|A|(s_1 - s_3)]^{\frac{1}{2}}.$$

Because $\mathscr{K}$, $\mathscr{E}$ depend only on one parameter, k, it is also convenient to parametrize E and L in terms of k and a further parameter, ρ, which is defined in terms of the roots as

$$\rho = \begin{cases} (s_2 - s_3)/s_3 & \text{if} \quad A > 0, \\ -k^2 s_3/s_2 & \text{if} \quad A < 0, \end{cases} \qquad 0 \leqslant \rho \leqslant \infty. \tag{4.9}$$

In terms of these variables we obtain ($k'^2 \equiv 1-k^2$):

$$\left.\begin{aligned} A>0\colon E &= \mu^2k^2\{(1+\rho)(\rho+2k^2)+\rho+k^2\}/AN^2,\\ L^2 &= -4\mu^3k^4(\rho+k^2)(1+\rho)/A^2N^3,\end{aligned}\right\} \tag{4.10a}$$

with $$N \equiv 3k^2+\rho(1+k^2);$$

$$\left.\begin{aligned} A<0\colon E &= \mu^2k^2k'^2\{(k^2-\rho)(1+\rho)-\rho k'^2\}/AN^2,\\ L^2 &= 4\mu^3k^4k'^4\rho(1+\rho)/A^2N^3,\end{aligned}\right\} \tag{4.10b}$$

with $$N \equiv k^2(1+\rho)+k'^2(k^2-\rho).$$

The roots s_1, s_2, s_3 can also be expressed in terms of (k, ρ, μ):

$$\left.\begin{aligned} A>0\colon s_1 &= -2\mu(\rho+k^2)/AN,\\ s_2 &= -2\mu k^2(1+\rho)/AN,\\ s_3 &= -2\mu k^2/AN,\end{aligned}\right\} \tag{4.11a}$$

$$\left.\begin{aligned} A<0\colon s_1 &= -2\mu k^2(1+\rho)/AN,\\ s_2 &= -2\mu k^2k'^2/AN,\\ s_3 &= 2\mu\rho k'^2/AN.\end{aligned}\right\} \tag{4.11b}$$

The persistence conditions $f = 0 = Lg$ may now be rewritten in terms of the new variables (k, ρ) and the unfolding parameters (μ, ν). This has the advantage that f and Lg become rational functions of ρ, k, $\Phi(k)$ and μ, ν, i.e. only one of the new variables, k, appears as an argument of the transcendental function

$$\Phi(k) \equiv \mathscr{E}(k)/\mathscr{K}(k). \tag{4.12}$$

Of course, instead of ρ, various other choices are possible for the second variable. In fact, any rational linear function of the roots will do the job. The choice (4.8) appeared to us most convenient. We make full use of this procedure in §§5 and 6.

4.4. *Stability considerations*

According to the averaging theorem, the stability properties of the solutions to the averaged equations (4.3) translate directly to the solutions of the original system (3.8). In the following sections we shall therefore address the stability of stationary solutions (E_0, L_0) of the system (4.3). Solutions with $L_0 = 0$ are the standing waves (SW), whereas those with $L_0 \neq 0$ are the modulated waves (MW). Their stability is described by the eigenvalues λ satisfying

$$\lambda^2-(f_E+g+2L_0^2g_{L^2})\lambda+f_Eg+2L_0^2(f_Eg_{L^2}-f_{L^2}g_E) = 0, \tag{4.13}$$

where f, g and the derivatives, indicated by subscripts, are to be evaluated on the solutions. Equation (4.13) becomes particularly simple for the standing waves ($L_0 = 0$), where the two eigenvalues are

$$\lambda_E = \partial f(E_0, 0)/\partial E, \quad \lambda_L = g(E_0, 0). \tag{4.14}$$

In the following, we have occasion to make use of the expression for $f_E(E_0, L_0)$. From (4.6) we find

$$f_E(E_0, L_0) = \frac{4}{3}\left(\nu - \frac{M\mu}{5A}\right) + \tfrac{4}{5}M\frac{J_1}{J_0} - \left[\frac{4}{3}\left(\nu - \frac{M\mu}{5A}\right)E_0 + (D - \tfrac{3}{5}M)\,L_0^2\right]\frac{J_0}{J_1}\frac{\partial}{\partial E}\frac{J_1}{J_0}. \tag{4.15}$$

We obtain a particularly simple form in the case $L_0 = 0$:

$$f_E(E_0, 0) = \nu + M(J_1/J_0). \tag{4.16}$$

5. The standing-wave solutions

The SW-solutions have zero angular momentum and thus break the SO(2)-symmetry. They are determined by the solutions E of

$$f(E, 0) = 0. \tag{5.1}$$

If we put $L = 0$ in (3.8) and consider only the equation for the amplitude, r, we recover the versal deformation of the Z(2)-equivariant normal form of a planar vector field with a nilpotent linear part that has already been analysed in detail (Knobloch & Proctor 1981). The only difference between the two is that here there are two eigenvalues that determine the stability of a closed orbit corresponding to a solution E of (5.1), whereas in the planar case only the sign of $\partial f/\partial E$ matters. Because of the reflection symmetry of (4.3), the two eigenvalues associated with a solution E of (5.1) are

$$\lambda_E \equiv \partial f(E, 0)/\partial E, \quad \lambda_L \equiv g(E, 0), \tag{5.2}$$

which we refer to as the E-eigenvalue and the L-eigenvalue, respectively. If $\lambda_E = 0$, we have a saddle-node (or limit-point) bifurcation, whereas for $\lambda_L = 0$, (4.3) undergoes a secondary bifurcation, i.e. an MW-solution branches off the SW-solution. Our discussion is conveniently divided into two parts.

5.1. *The case $A > 0$*

In this case, the SW-solutions are always oscillations about the trivial solution, and occur for $\mu < 0$ and $0 = s_3 < s < s_2$ (see figure 2). The condition $s_3 = 0$ implies $\rho = \infty$ (cf. (4.8)) and yields

$$As_1 = -\frac{2\mu}{1+k^2}, \quad As_2 = -\frac{2\mu k^2}{1+k^2}, \quad AE = \frac{\mu^2 k^2}{(1+k^2)^2}. \tag{5.3}$$

At leading order in ϵ, the SW-solution satisfies

$$r'^2 = \tfrac{1}{2}A(s_2 - r^2)(s_3 - r^2),$$

which has the one-parameter family of solutions given in terms of an elliptic function of modulus k by

$$r^2 = s_2\, sn^2\theta, \quad \theta = (\tfrac{1}{2}As_1)^{\frac{1}{2}}\tau.$$

To relate k to the other parameters we use the persistence condition (5.1), which we express in terms of μ, ν and k by eliminating s_1, s_2 and E between (4.6), (4.7) and (5.3). We obtain

$$5A\nu = 2M\mu R_1(k), \tag{5.4}$$

where

$$R_1(k) = \frac{k'^2(1+k'^2) - 2(k'^2+k^4)\,\Phi(k)}{(1+k^2)\,[k'^2 - (1+k^2)\,\Phi(k)]}. \tag{5.5}$$

For small k, the amplitude of the solutions is also small. Because $R_1(k) = \frac{5}{4}k^2 + O(k^4)$ for $k^2 \ll 1$, we see that the SW-solutions bifurcate from the trivial solution along the line $H_0(\nu = 0, \mu < 0)$, and into the region $M\nu < 0, \mu < 0$. On the other hand, as $k \to 1-$, the period of the oscillations tends toward infinity, and the SW-solutions undergo a saddle-loop bifurcation. Because $\Phi(k) \sim 1/\ln(4/k') + O(k')$, $k' \ll 1$, we see that $R_1(k) \to \frac{1}{2}$ as $k \to 1-$, and hence the saddle-loop bifurcation occurs on the line

$$SL_S: 5A\nu = M\mu, \quad \mu < 0. \tag{5.6}$$

As k increases, the amplitude of the oscillations increases monotonically, and the SW-branch terminates with the formation of a heteroclinic orbit connecting the saddles $r = \pm\sqrt{s_1} = \pm\sqrt{s_2}$. Thus the SW-solutions exist in the region of the (μ, ν)-plane bounded by the lines H_0 and SL_S.

The standing-wave solution encounters a stability exchange when the L-eigenvalue changes sign, i.e. when $g(E, 0) = 0$. In terms of k, $g(E, 0)$ is given by

$$g(E,0) = \nu - \frac{2D}{A}\mu\left(\frac{1-\Phi(k)}{1+k^2}\right). \tag{5.7}$$

Setting $g = 0$ and by using (5.4) shows that the L-instability occurs on the half-line

$$L_S: A\nu = \frac{2D\mu(1-\Phi(k))}{1+k^2}, \quad \mu < 0, \tag{5.8}$$

where k satisfies

$$\frac{D}{M} = R_2(k) \equiv \frac{k'^2(1+k'^2) - 2(k'^2+k^4)\,\Phi(k)}{5(1-\Phi(k))\,[k'^2 - (1+k^2)\,\Phi(k)]}. \tag{5.9}$$

The function $R_2(k)$ decreases monotonically from $\frac{1}{2}$ to $\frac{1}{5}$ as k increases from 0 to 1. Hence, the L-instability is present only if $\frac{1}{5} < D/M < \frac{1}{2}$. This bifurcation produces the branch of modulated waves.

The E-eigenvalue is easily found from (4.16):

$$\left.\begin{aligned} f_E(E,0) &= \nu - \frac{2M}{A}\mu\left(\frac{1-\Phi(k)}{1+k^2}\right) \\ &= \frac{2M\mu}{5A}\left[R_1(k) - 5\frac{1-\Phi(k)}{1+k^2}\right], \end{aligned}\right\} \tag{5.10}$$

because λ_E must be evaluated on the SW-branch, i.e. on (5.4). The term in the square brackets is always negative (for $k^2 > 0$), so that λ_E has the same sign as M, and an E-instability does not occur.

We now turn to the stability assignments. We first observe that the SW-solutions exist for (μ, ν) in the region bounded by the lines H_0 and SL_S, with $M\nu < 0, \mu < 0$. It is readily shown from (5.7) and (5.4) that

$$\lambda_L = \nu(1 - 2D/M) + O(k^2), \quad k^2 \ll 1. \tag{5.11}$$

Hence, we have the assignments $(\operatorname{sgn}\lambda_E, \operatorname{sgn}\lambda_L) = (\operatorname{sgn} M, \operatorname{sgn} M)$ for $\frac{1}{2} < D/M$; $(\operatorname{sgn} M, -\operatorname{sgn} M)$ between the lines H_0, L_S, and $(\operatorname{sgn} M, \operatorname{sgn} M)$ between L_S and SL_S when $\frac{1}{5} < D/M < \frac{1}{2}$; and finally $(\operatorname{sgn} M, -\operatorname{sgn} M)$ when $D/M < \frac{1}{5}$.

5.2. *The case* $A < 0$

From figure 2 we infer that SW-solutions occur in two different ways, i.e. we can have oscillations about the trivial solution (zero mean) and about the non-trivial steady state (non-zero mean). The former arise if $s_2 = 0$ implying $\rho = \infty$, and the latter occur for $s_3 = 0$ implying $\rho = 0$. We denote these two types of SW-solutions by SW_2 and SW_3, respectively, and treat them separately.

(*a*) *The* SW_2*-branch*

The condition $\rho = \infty$ yields the following expressions for the roots s_1, s_3 and the energy,

$$As_1 = -\frac{2\mu k^2}{k^2-k'^2}, \quad As_3 = \frac{2\mu k'^2}{k^2-k'^2}, \quad AE = -\frac{\mu k^2 k'^2}{(k^2-k'^2)^2}, \tag{5.12}$$

where $\mu(k^2-k'^2) \geqslant 0$. At leading order in ϵ, the SW_2-solution satisfies $r'^2 = \frac{1}{2}A(s_1-r^2)(s_3-r^2)$ with the solution

$$r^2 = s_1\, cn^2\theta, \quad \theta = [\tfrac{1}{2}A(s_3-s_1)]^{\frac{1}{2}}\tau.$$

Substituting (5.12) into (4.6–4.11) allows us to express the persistence condition in terms of k. We obtain

$$5A\nu = -2M\mu R_3(k), \tag{5.13}$$

where

$$R_3(k) \equiv \frac{k'^2(1+k'^2)-2(k'^2+k^4)\,\Phi(k)}{(k^2-k'^2)\,[k'^2+(k^2-k'^2)\,\Phi(k)]}. \tag{5.14}$$

For small k, $R_3 \sim \frac{5}{4}k^2$, so that SW_2 branches off the trivial solution on H_0 into the region $M\nu < 0$. We note that $R_3(k) \to \pm\infty$ if $k^2 \to \frac{1}{2}\mp$, i.e. $\mu = 0$ along SW_2 for $k^2 = \frac{1}{2}$.

Proceeding as in 5.1 we find that the L-eigenvalue is given by

$$\lambda_L = \nu + \frac{2D}{A}\mu\left(\frac{k'^2-\Phi(k)}{k^2-k'^2}\right)$$

with an L-instability along the line

$$L_{S2}\colon A\nu = 2D\mu\left(\frac{\Phi(k)-k'^2}{k^2-k'^2}\right), \quad \mu < 0, \tag{5.15}$$

where k satisfies

$$\frac{D}{M} = R_4(k) \equiv \frac{-k'^2(1+k'^2)+2(k'^2+k^4)\,\Phi(k)}{5(\Phi(k)-k'^2)\,[k'^2+(k^2-k'^2)\,\Phi(k)]}. \tag{5.16}$$

Inspection shows that $R_4(k)$ increases monotonically from $\frac{1}{2}$ to ∞ as k varies from 0 to 1; hence the L-instability is present for $D/M > \frac{1}{2}$. For small k, λ_L is again given by (5.11).

The E-eigenvalue, λ_E, takes the form

$$\left.\begin{aligned} AC\lambda_E &= A\nu - \frac{2M\mu(\Phi(k)-k'^2)}{k^2-k'^2} \\ &= -\frac{2M\mu}{k^2-k'^2}[\tfrac{1}{5}(k^2-k'^2)\,R_3(k)+\Phi(k)-k'^2]. \end{aligned}\right\} \tag{5.17}$$

For small k^2, the term in the square brackets behaves as $\frac{3}{4}k^2$, so that $M\lambda_E > 0$ near $k = 0$. An E-instability (saddle-node bifurcation) appears when the term in the square brackets vanishes.

This occurs when $k^2 \approx 0.93$ (Knobloch & Proctor 1981). Consequently, there is a saddle node on the SW_2-branch along the half-line

$$SN_{S2}: A\nu = \frac{2M\mu(\Phi(k)-k'^2)}{k^2-k'^2}, \quad \mu > 0, \quad k^2 \approx 0.93, \tag{5.18}$$

or

$$A\nu \approx 0.74M\mu, \quad \mu > 0.$$

Finally, we consider infinite-period bifurcations. Because $R_3(k) \to -2$ as $k \to 1-$, the homoclinic (actually, ambiclinic) orbits lie on the line

$$SL_S: 5A\nu = 4M\mu, \quad \mu > 0. \tag{5.19}$$

This line marks the smooth transition from the SW_2-branch to a branch of non-zero-mean standing waves, SW_3.

The stability assignments are as follows: $\operatorname{sgn}\lambda_E = \operatorname{sgn} M$ between H_0 and SN_{S2}, and $-\operatorname{sgn} M$ between SN_{S2} and SL_S; $\operatorname{sgn}\lambda_L = -\operatorname{sgn} M$ if $D/M < \frac{1}{2}$, whereas if $D/M > \frac{1}{2}$, $\operatorname{sgn}\lambda_L = \operatorname{sgn} M$ between H_0 and L_{S2}, and $-\operatorname{sgn} M$ between L_{S2} and SL_S.

(*b*) *The* SW_3*-solution*

We now describe the branch of non-zero-mean standing waves for which $s_3 = 0$, $\rho = 0$, and

$$As_1 = -\frac{2\mu}{1+k'^2}, \quad As_2 = -\frac{2\mu k'^2}{1+k'^2}, \quad AE = \frac{\mu^2 k'^2}{(1+k'^2)^2}, \quad \mu > 0. \tag{5.20}$$

The SW_3-oscillations satisfy $r'^2 = \frac{1}{2}A(r^2-s_1)(r^2-s_2)$ with the one-parameter family of solutions

$$r^2 = s_1\,\mathrm{dn}^2\theta, \quad \theta = (-\tfrac{1}{2}As_1)^{\frac{1}{2}}\tau.$$

Proceeding as before, we obtain the following persistence condition,

$$5A\nu = 2M\mu R_5(k), \quad \mu > 0, \tag{5.21}$$

where

$$R_5(k) \equiv \frac{k'^2(1+k'^2)-2(k'^2+k^4)\,\Phi(k)}{(1+k'^2)\,[2k'^2-(1+k'^2)\,\Phi(k)]}. \tag{5.22}$$

Near $k = 0$, $R_5 \sim \frac{5}{2}-\frac{5}{64}k^4+O(k^6)$. Consequently, (5.21) approaches the line L_M (equation (3.17)) as $k \downarrow 0$, and the SW_3-branch bifurcates from the nontrivial steady-state branch into the region to the left (right) of L_M if $M < 0$ ($M > 0$).

The L-eigenvalue is given by

$$\lambda_L = \nu - \frac{2D}{A}\mu\frac{\Phi(k)}{1+k'^2}, \tag{5.23}$$

and hence the L-instability occurs on the half-line

$$L_{S3}: A\nu = 2D\mu\frac{\Phi(k)}{1+k'^2}, \quad \mu > 0, \tag{5.24}$$

where k satisfies

$$\frac{D}{M} = R_6(k) \equiv \frac{k'^2(1+k'^2)-2(k'^2+k^4)\,\Phi(k)}{5\Phi(k)\,[2k'^2-(1+k'^2)\,\Phi(k)]}. \tag{5.25}$$

Because $R_6(k)$ increases monotonically from 1 to ∞ as k^2 varies from 0 to $1-$, the L-instability, i.e. the bifurcation of MW from SW_3, is present only if $D/M > 1$. Moreover, for small k^2, we find that $g = \nu(1-D/M)+O(k^2)$. A further calculation shows that the SW_3-branch does not undergo an E-instability (a saddle-node bifurcation), and that it joins smoothly to

SW_2 along the line SL_S in an infinite period bifurcation. The sign of the E-eigenvalue is always given by $-\operatorname{sgn} M$. The stability assignments are therefore as follows: $(\operatorname{sgn}\lambda_E, \operatorname{sgn}\lambda_L) = (-\operatorname{sgn} M, -\operatorname{sgn} M)$ if $D/M < 1$; $(-\operatorname{sgn} M, \operatorname{sgn} M)$ between the lines L_M and L_{S3}; and $(-\operatorname{sgn} M, -\operatorname{sgn} M)$ between L_{S3} and SL_S when $D/M > 1$.

This completes our discussion of the standing-wave branches.

6. The modulated-wave solutions

The modulated waves correspond to stationary solutions of the averaged equations with non-zero angular momentum. They are determined by the solutions (E, L) of the pair of equations

$$f(E, L^2) = 0, \quad g(E, L^2) = 0. \tag{6.1}$$

The most convenient way to solve (6.1) is in terms of the variables (k, ρ) introduced in §2. We first substitute $\nu = -DJ_1/J_0$ obtained from $g = 0$ into f, and express E, L and the roots s_1, s_2, s_3 in terms of k and ρ. Then $f = 0$ reduces to an equation in (k, ρ) alone with the remarkable feature that it depends linearly on ρ. This allows one easily to express ρ in terms of k. We obtain

$$\frac{\rho}{k^2} = \frac{-5D[k'^2 - 2(1+k'^2)\,\Phi(k) + 3\Phi^2(k)]}{5Dk'^2 - Mk'^2(1+k'^2) + 2[M(k'^2+k^4) - 5D]\,\Phi(k) + 5D(1+k^2)\,\Phi^2(k)} \tag{6.2a}$$

for $A > 0$, and

$$\frac{\rho}{k^2} = \frac{-Mk'^2(1+k'^2) + 2[M(k'^2+k^4) + 5Dk'^2]\,\Phi(k) - 5D(1+k'^2)\,\Phi^2(k)}{Mk'^2(1+k'^2) - 5Dk'^4 + 2[5Dk'^4 - M(k'^2+k^4)]\,\Phi(k) + 5D(k^2-k'^2)\,\Phi^2(k)} \tag{6.2b}$$

for $A < 0$. Now we are in the position to relate k to the unfolding parameters μ, ν by inserting (6.2) into $\nu = -DJ_1/J_0$. This gives the following persistence condition for the MW-solutions:

$$A\nu = \frac{2D\mu(M - 15DQ(k))}{3M - 5D}, \tag{6.3}$$

where

$$Q(k) = \frac{\Phi(k)\,(k'^2 - \Phi(k))\,(1 - \Phi(k))}{k'^2(1+k'^2) - 2(k'^2+k^4)\,\Phi(k)}. \tag{6.4}$$

We note that (6.3), (6.4) hold for both cases, $A > 0$ and $A < 0$. Equations (6.2)–(6.4) provide a parametric representation for the MW-branches. Of course, so that an MW-solution can exist, the unperturbed system ($\epsilon = 0$) must exhibit closed orbits with $L \neq 0$. For $A > 0$ this restricts μ to negative values but imposes no *a priori* restriction for $A < 0$.

The function $Q(k)$ decreases monotonically from $\frac{2}{15}$ to 0 when k increases from 0 to 1 (see Appendix B). For small k, $Q(k) = \frac{2}{15} - k^4/320 + O(k^6)$, which yields

$$A\nu = \frac{2D\mu}{3M - 5D}\{M - 2D + \tfrac{3}{64}Dk^4 + O(k^6)\} \tag{6.5}$$

for the local form of the persistence condition, and ρ behaves as

$$\rho = \frac{Dk^2}{M - 2D} + O(k^4) \quad \text{for} \quad A > 0, \tag{6.6a}$$

$$\rho = -\frac{M - D}{M - 2D}k^2 + O(k^4) \quad \text{for} \quad A < 0. \tag{6.6b}$$

If we substitute (6.6) into L^2, (equation 4.9)), and then let $k^2 \downarrow 0$ we find, regardless of the sign of A,

$$L^2 = -\frac{4\mu^3}{A^2}\frac{(M-D)(M-2D)^2}{(3M-5D)^3} + O(k^2), \tag{6.7}$$

i.e. the value that L^2 takes on the TW-branch along the line L_H, (3.22). Thus we recover the Hopf bifurcation on L_H where the MW-solution branches off the TW-solution. Obviously (because $L^2 \geqslant 0$, $\rho \geqslant 0$) the sign of μ on MW must be such that the persistence condition (6.3) defines a family of half-lines, parametrized by k, that tends to L_H for $k \downarrow 0$. The region into which MW bifurcates from TW is easily deduced from (6.4) because $k^2 \geqslant 0$. This information and a well-known theorem (Sattinger 1973) about the stability exchange at a Hopf bifurcation tells us the stability of the MW-branch near L_H: If $AM\mu < 0$ and $AM\mu > 0$, MW bifurcates from TW into the region obtained by clockwise and counterclockwise rotation of L_H, respectively, with stability assignments (sgn M, $-$sgn $(M\mu)$). Recall, however, that this Hopf bifurcation is only present for $0 < D/M < \frac{1}{2}$ if $A > 0$ and for $\frac{1}{2} < D/M < 1$ if $A < 0$. We deduced these restrictions in §3; here they follow from the conditions $\rho \geqslant 0$, $L^2 \geqslant 0$. Note that $\rho \downarrow 0$ as $k \downarrow 0$, but L^2 tends to a positive value in the limit. This reflects the fact that the transformation $(k, \rho, \mu) \rightarrow (E, L^2, \mu)$ becomes singular for $k \downarrow 0$.

Because $Q(k)$ never becomes infinite, the sign of μ cannot change on the MW-branch. Furthermore, because of the monotonic behaviour of $Q(k)$, there is only one single MW-branch, i.e. saddle nodes do not exist. From these facts we conclude that there must be precisely two half-lines lying in a half-plane corresponding to a fixed sign of μ between which the MW-branch exists. On these lines, MW must be created in a secondary bifurcation. There are three possibilities: a Hopf bifurcation from TW, a secondary bifurcation from an SW-branch (pitchfork-type bifurcation of a solution of (6.1) with $L \neq 0$ from a solution with $L = 0$), or an infinite-period bifurcation where the frequency of the periodic orbit in (r, r', L)-space corresponding to the solution (E, L) of (6.1) tends to zero, thereby producing a homoclinic orbit of a hyperbolic stationary point, i.e. of a TW-solution whose r-coordinate corresponds to a local maximum of the potential $V(r)$. The last transition is, however, only possible if $A > 0$ because the potential has no maximum if $A < 0$.

General stability calculations along MW are very tedious; therefore, we confine ourselves to analysing local stabilities near the two half-lines of secondary bifurcations which, as explained before, must lie in a half-plane given by a particular sign of μ. Because saddle nodes are not present along MW, only stability exchanges of the type $(++) \leftrightarrow (--)$ are possible; these give rise to a tertiary Hopf bifurcation in the averaged equations and produce a three-torus if the decoupled phase, ϕ, is included. If the assignment is $(+-)$ near one of the two secondary-bifurcation lines, it must be preserved along the whole branch. If both signs in the stability assignment are equal and the same at both ends of the branch, we shall make the plausible conjecture that no tertiary bifurcation occurs; if they are opposite we conclude that a tertiary Hopf bifurcation (in fact, an odd number) occurs. It will, however, turn out that the assignments are always equal near both ends, i.e. there is no sufficient condition for a tertiary Hopf bifurcation.

We have already computed all the possible half-lines leading to secondary bifurcations of the MW-branch in §§3 and 5, apart from the global (homoclinic) bifurcation in the case $A > 0$. This calculation is very easy because we only need to consider the limit $k \rightarrow 1-$ in the persistence

conditions (6.2, 6.3). As $k \to 1-$, the function $Q(k)$ behaves as $\frac{1}{2}\Phi + o(\Phi)$ and the condition (6.3) becomes

$$A\nu = \frac{2D\mu}{3M-5D}\{M - \tfrac{15}{2}D\Phi + o(\Phi)\} \quad (k \to 1-). \tag{6.8}$$

For ρ we obtain

$$\rho = \frac{5D}{M-5D}\left\{1 - \frac{3M-5D}{2(M-5D)}\Phi + o(\Phi)\right\}. \tag{6.9}$$

Equation (6.8) shows that a homoclinic bifurcation appears on the half-line

$$SL_M\colon A\nu = \frac{2DM\mu}{3M-5D}, \quad \mu < 0, \tag{6.10}$$

provided that $0 < D/M < \frac{1}{5}$ (because $\rho \geqslant 0$). In Appendix C the stability assignment near SL_M is computed. It turns out to be (sgn M, sgn M).

We are now in the position to give a complete picture of the MW-branch. We do this for the two cases $A > 0$ and $A < 0$ separately.

(*a*) *The case* $A > 0$

The Hopf bifurcation on L_H exists if $0 < D/M < \frac{1}{2}$ and the homoclinic transition is present for $0 < D/M < \frac{1}{5}$. In §5 it was shown that the SW-branch undergoes an L-instability if $\frac{1}{5} < D/M < \frac{1}{2}$, giving rise to a secondary bifurcation of MW from SW. We conclude that the MW-branch can only exist in the parameter range $0 < D/M < \frac{1}{2}$. One line of secondary bifurcations is always given by L_H. The other one corresponds to a homoclinic transition if $0 < D/M < \frac{1}{5}$ and to a secondary bifurcation from SW if $\frac{1}{5} < D/M < \frac{1}{2}$. Because we know the region into which MW bifurcates from L_H, the other half-line of secondary bifurcations must be located between L_H and the negative (positive) ν-axis for $M > 0$ ($M < 0$). The assignment of MW near SL_M is (sgn M, sgn M). If we follow the SW-branch on a path parallel to the μ-axis in the region $\mu < 0$, starting at $\mu = 0$, then there is a supercritical bifurcation of MW from SW on L_S if $\frac{1}{5} < D/M < \frac{1}{2}$. From this we deduce the assignment (sgn M, sgn M) for MW near L_S. In summary, the assignment near both ends of MW is always (sgn M, sgn M), i.e. tertiary Hopf bifurcations can only occur in pairs.

(*b*) *The case* $A < 0$

Here we have the Hopf bifurcation from TW on L_H for $\frac{1}{2} < D/M < 1$, the L-instability of SW_2 for $D/M > \frac{1}{2}$ and another L-instability of SW_3 for $D/M > 1$, i.e. half-lines of secondary bifurcations exist pairwise in the parameter range $\frac{1}{2} < D/M < \infty$. If $\frac{1}{2} < D/M < 1$, we have the Hopf bifurcation on L_H that allows us to draw similar conclusions as in the case $A > 0$. From the results of §§3 and 4, and the local behaviour of MW near L_H already described, we find that L_{S_2} is below (above) L_H if $\frac{1}{2} < D/M < \frac{3}{5}$ ($\frac{3}{5} < D/M < 1$), and both half-lines are in the region $M\nu < 0$, $(D/M - \frac{3}{5})\mu > 0$. The MW-branch exists between L_H and L_{S_2}, and the assignments near both ends are (sgn M, $-$sgn $(M\mu)$). If D/M passes through 1, the Hopf bifurcation on L_H for $D/M < 1$ disappears and is replaced by an L-instability on the SW_3-branch for $D/M > 1$. This does not, however, affect the L-instability of SW_2. Consequently, if $D/M > 1$, the half-line L_{S_2} is located in the region $\mu > 0$, $M\nu < 0$. The MW-solution bifurcates from L_{S_2} into the region obtained by clockwise (counterclockwise)

rotation of L_{S2} for $M < 0$ ($M > 0$), until it terminates on L_{S3}, which is located below L_{S2}. The assignment of MW near L_{S2} is still (sgn M, $-$sgn M). Because saddle nodes do not exist, this assignment must be preserved on the whole branch. Consequently, there are no necessary conditions for a tertiary Hopf bifurcation in the averaged equations.

This concludes our discussion of the MW-solutions.

7. Stability diagrams and bifurcation diagrams

In the preceding sections we have described the different types of non-trivial solutions admitted by the normal form (3.3); in this section we bring all the pieces of information together. We represent the results in two complementary ways. In the stability diagrams we indicate how the solutions depend on the unfolding parameters μ, ν, whereas the bifurcation diagrams show their dependence on a distinguished bifurcation parameter.

The plane of the unfolding parameters (μ, ν) is divided into a number of regions by the half-line H_0, and the line L_0, along which occur, respectively, the primary bifurcations to the TW-solutions and SW-solutions, and to the steady states, and the half-lines, L_m, L_M, L_S etc., along which secondary bifurcations take place. In each region of the (μ, ν)-plane, a certain configuration of non-trivial solutions with specific stability assignments occurs that changes if we pass into a neighbouring region. Crossing a line of primary or secondary bifurcations gives rise to the appearance or disappearance of a non-trivial solution. In the case of local bifurcations (Hopf or pitchfork type), the birth of a new solution is always combined with a stability exchange of another solution from which the former branches off.

To summarize the results of §§3–6, we sketch the lines of primary and secondary bifurcations in the (μ, ν)-plane. The existence domain of a particular non-trivial solution branch is marked by a curve passing through it and connecting the two bifurcation lines where the solution is created. Each curve is divided into pieces, separated by arrows, on which the branch possesses a particular stability assignment (see figures 4 and 7 below). The resulting figure is called a *stability diagram*. The trivial solution, T, is not included in the stability diagrams. Recall from §3 that the T-solution exists for all values of (μ, ν) with assignments $(+-)$ in $\{\mu > 0\}$ and (sgn ν, sgn ν) in $\{\mu < 0\}$. For the SS-branch, the first two signs in the assignment correspond to the (r, r')-eigenvalues and the third to the L-eigenvalue. Similarly, the assignment for the SW-branch is (sgn λ_E, sgn λ_L). There is no such natural distinction between the eigenspaces in the case of TW and MW.

For a fixed sign of A the stability diagrams change if we pass through certain straight lines in the (D, M)-plane. Our analysis is not valid on these lines, i.e. we have to impose a number of *non-degeneracy conditions* of the form $D/M \neq c$. Violation of one of the non-degeneracy conditions gives rise to the coalescence of two or more bifurcation lines in the unfolding plane (more precisely, their slopes at the origin because higher-order terms may prevent the lines from being straight). If we pass through a critical value of D/M, then either an event may occur where two or more independent bifurcation phenomena happen simultaneously, or our normal form degenerates into a bifurcation of co-dimension three. In the former case, two or more (μ, ν)-lines simply cross each other, i.e. their relative position changes. The latter case is always combined with the appearance or disappearance of at least one line of secondary bifurcations.

We now discuss the two sign cases $A > 0$ and $A < 0$ in turn.

7.1. *The case* $A > 0$

There are four non-degeneracy conditions, namely, $D \neq 0$, $M \neq 0$ and $D/M \neq \frac{1}{2}$, $D/M \neq \frac{1}{5}$ that divide the (D, M)-plane into eight regions, as shown in figure 3. The regions are enumerated according to I$\pm$–IV$\pm$ where '+' or '−' indicates the sign of M. Regions with the same roman numeral but opposite signs are complementary to each other, i.e. all the lines of secondary

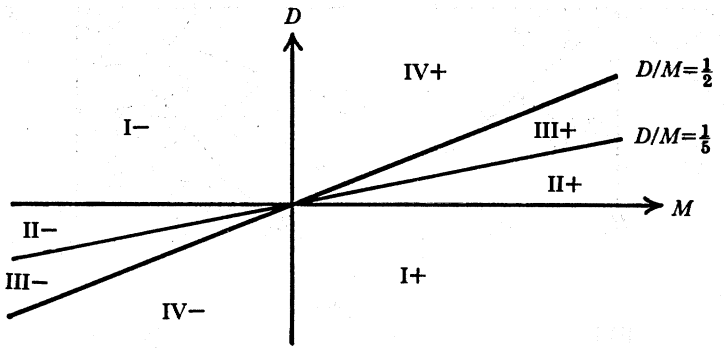

FIGURE 3. Division of (D, M)-plane for $A > 0$.

bifurcations and the regions between them are reflected at the μ-axis and the signs in the assignments of the non-trivial solutions are reversed. Therefore, we may confine ourselves to a description of the case $M > 0$. The stability diagrams corresponding to regions I$\pm$ to IV$\pm$ are shown in figure 4. Before turning to the individual stability diagrams, we first recall the meaning of the various half-lines giving rise to secondary bifurcations:

L_m: TW bifurcates from SS,
SL_S: heteroclinic bifurcation of SW,
L_H: Hopf bifurcation of MW from TW,
L_S: secondary bifurcation of MW from SW,
SL_M: homoclinic bifurcation of MW.

In the stability diagram I+, no MW-solution exists. There are only the two secondary bifurcation lines L_m and SL_S, which are located on opposite signs of H_0, i.e. SW and TW do not exist simultaneously. When passing from region I+ to region II+, the line L_m crosses H_0 and lies now between SL_S and H_0, thereby throwing off L_H and L_S and leading to the appearance of a MW-branch. Approaching the line $D/M = \frac{1}{5}$ from II+, SL_S, L_m and SL_M coalesce and disappear in region III+ where the role of the other end for MW is taken over by L_S. Simultaneously, L_m and SL_S have changed their relative position, i.e. SL_S is now below L_m. Finally, passing through $D/M = \frac{1}{2}$, L_S and L_H coalesce on H_0 and disappear in region IV+ where no MW-branch is present. If $M = 0$, SL_S and H_0 coincide. We note that all the lines in figure 3 correspond to degeneracies of co-dimension three.

In figure 5 we have displayed the stability diagrams of figure 4 schematically in terms of bifurcation diagrams. These bifurcation diagrams are obtained by following a straight path through the (μ, ν)-plane that starts in the region $\{\mu < 0, \nu < 0\}$ and hits all lines of secondary bifurcations as well as H_0 and, at a positive value of ν, the line L_0. Local and global bifurcations

are indicated by dots and small circles, respectively. Note that a global secondary bifurcation does not induce a stability exchange on the branch of 'equilibria' (TW or SS) on which the closed orbit (MW or SW) terminates. The choice of our path implies that the trivial solution is always stable to the left of the first primary bifurcation and that the steady state bifurcates subcritically.

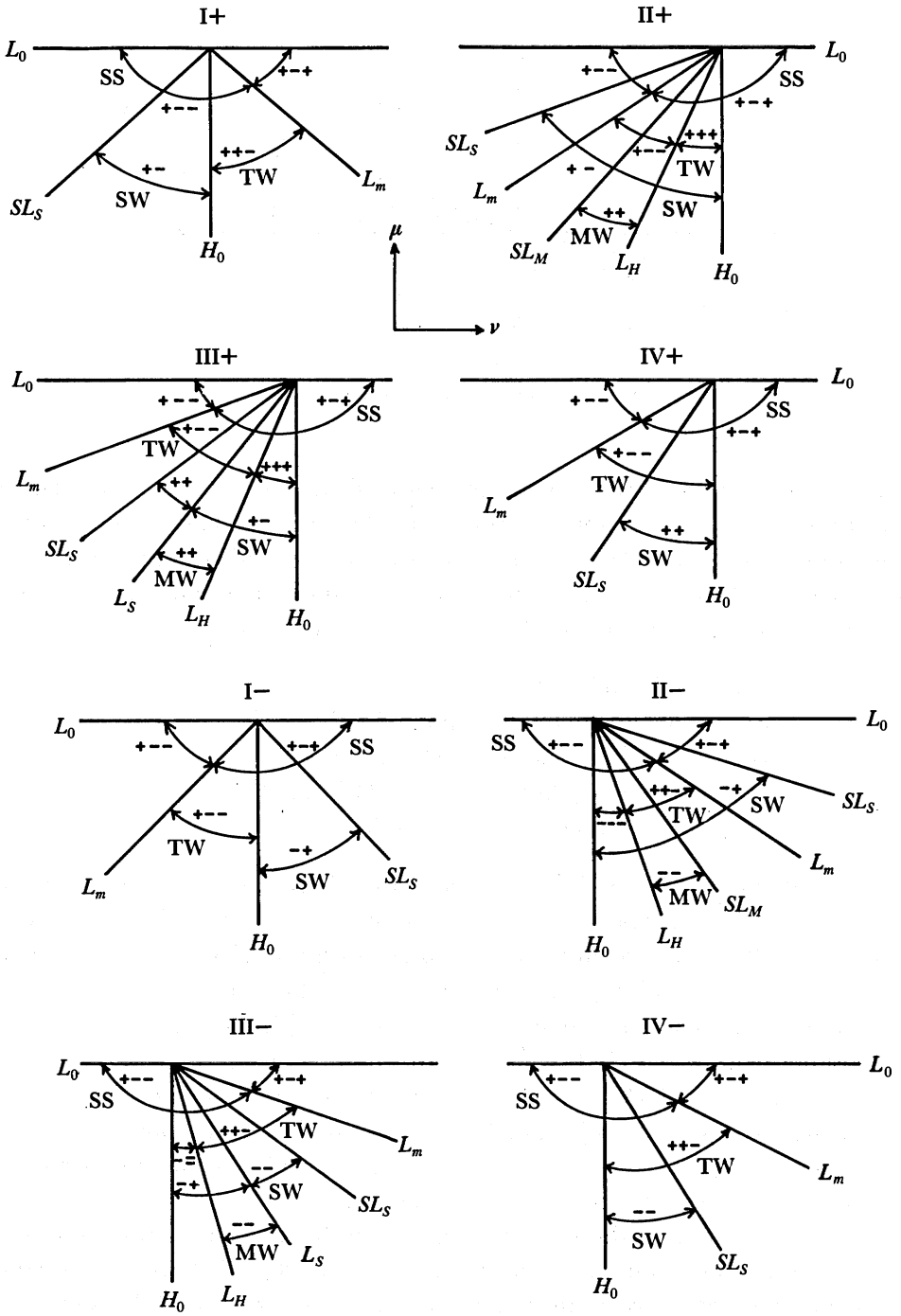

FIGURE 4. Stability diagrams corresponding to the regions marked in figure 3.

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7.2. *The case* $A < 0$

For $A < 0$, the (D, M)-plane is divided into 18 different regions by 9 lines through the origin as shown in figure 6. The corresponding non-degeneracy conditions are again $D \neq 0$, $M \neq 0$ and $D/M \neq c$ where $c = \frac{1}{2}, \frac{3}{5}, c_1 \approx 0.70, c_2 \approx 0.74, \frac{3}{4}, \frac{4}{5}, 1$. The regions in the (D, M)-plane where these conditions are satisfied are enumerated according to $\mathrm{I}\pm, \mathrm{II}\pm, \ldots, \mathrm{IX}\pm$ with the same meaning of the signs '+' and '−' as in the case $A > 0$, i.e. reflection of the v-axis and conversion of the stability assignments maps complementary stability diagrams onto each other. Recall from §5 that the SW-solutions are now distinguished according to oscillations about $r = 0$ (SW_2) and oscillations about the non-trivial steady state (SW_3).

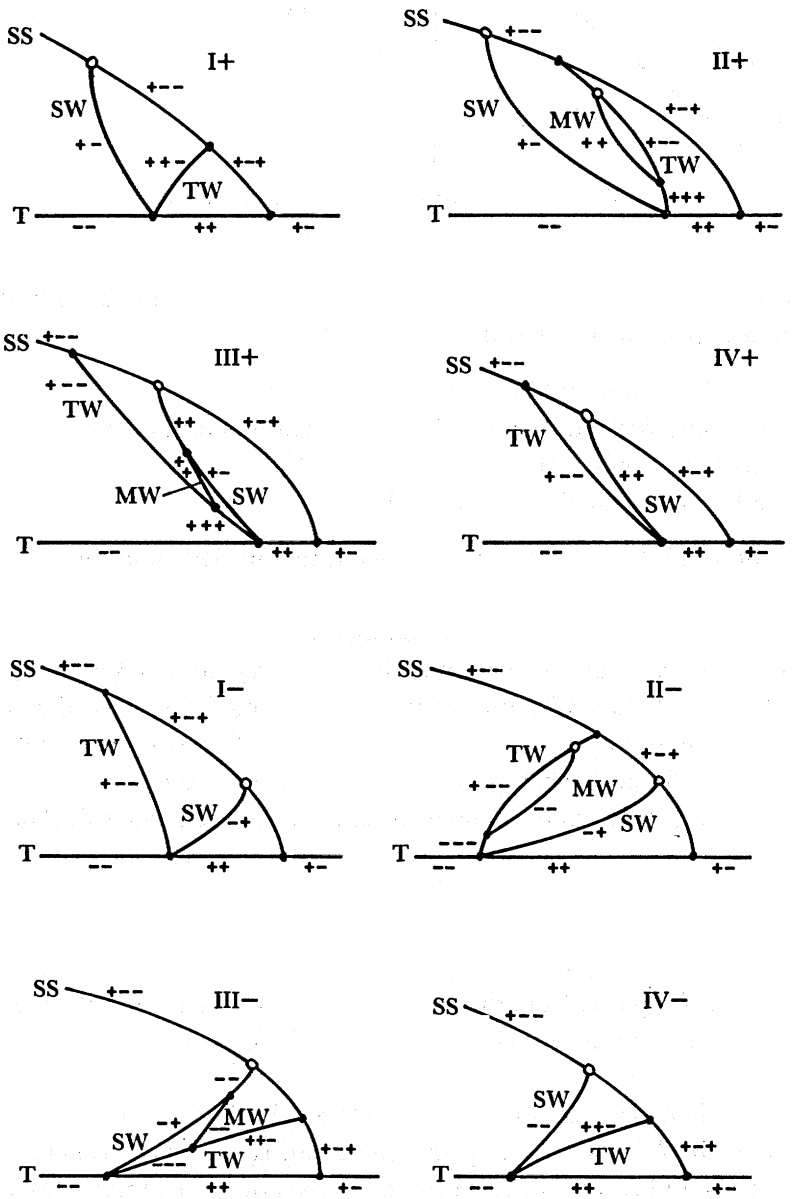

FIGURE 5. Bifurcation diagrams corresponding to the regions marked in figure 3.

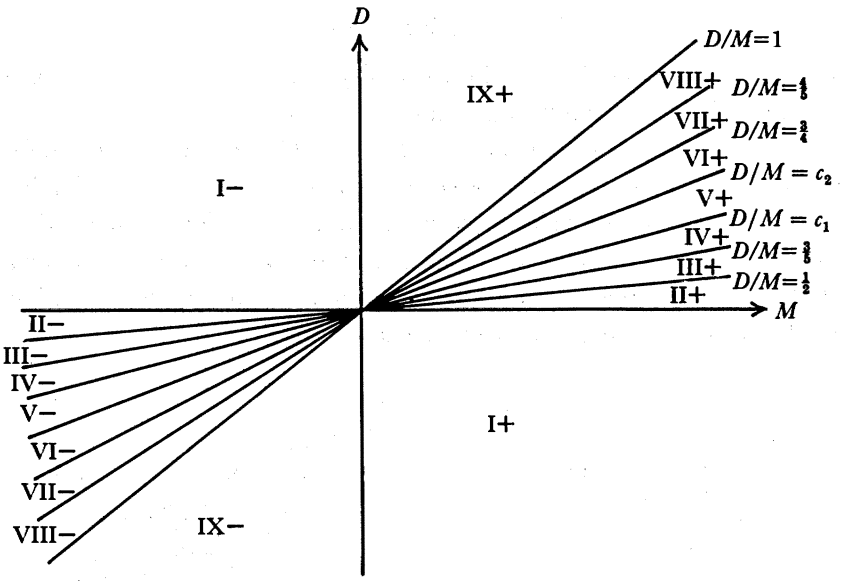

FIGURE 6. Division of (D, M)-plane for $A < 0$.

In addition to L_m and L_H, which have the same meaning as for $A > 0$, the following secondary bifurcations occur for $A < 0$

L_M:	SW_3 branches off SS,
SL_S:	SW_3 and SW_2 undergo a global bifurcation and join smoothly to each other,
$L_{Sj} (j = 2, 3)$:	secondary bifurcation of MW from SW_j,
SN_{S2}:	saddle node for the SW_2-branch.

The presence of a saddle node implies that two different SW_2-solutions exist between SN_{S2} and SL_S. Both are created along SN_{S2}, but one of them joins the SW_3-branch along SL_S whereas the other one undergoes a primary bifurcation at H_0.

In figure 7 the stability diagrams corresponding to the regions marked in figure 6 are shown. The existence domain of the SW-solutions is characterized by a curve that runs from H_0 to SN_S and returns from there to L_M after crossing SL_S. The two types SW_2 and SW_3 are distinguished by the full and broken parts of the curve, respectively.

On the lines in the (D, M)-plane shown in figure 6, the following degeneracies occur for $M \geqslant 0$.

(i) $M = 0$, $D < 0$: the lines L_M, L_{S2}, SL_S, L_{S3}, SN_{S2} coalesce with the positive μ-axis. All of them as well as the MW-branch exist in region IX−, but the two secondary bifurcation lines L_{S2}, L_{S3} disappear in region I+, together with MW.

(ii) $D = 0$; the line L_m passes from the region $\mu > 0, \nu > 0$ (I+) into the region $\mu > 0, \nu < 0$ (II+). This transition is combined with a reflection of the existence domain of TW, but does not produce a MW-branch.

(iii) $D/M = \frac{1}{2}$: When passing from II+ to III+, the line H_0 throws off both the Hopf bifurcation L_H and the L-instability L_{S2}, thereby creating the MW-branch.

(iv) $D/M = \frac{1}{5}$: L_H and L_{S2} coalesce with the left part of H_0. In region III+, L_H lies above

L_{S_2} and MW has the assignment $(++)$. In region IV+, the relative position of L_H and L_{S_2} is inverted resulting in the new assignment $(+-)$ for MW.

(v) $D/M = c_1 \approx 0.70$: L_M and L_{S_2} cross each other. The number c_1 is given by $c_1 = R_4(k)$ where k satisfies $(k^2-k'^2)/(\Phi(k)-k'^2) = 2R_4(k)$, or

$$\Phi(k) = \frac{(1-8k^2)(1-k^2)}{1-16k^2+16k^4},$$

which has a unique solution in $0 < k < 1$, given by $k^2 \approx 0.71$.

(vi) $D/M = c_2 \approx 0.74$; L_m and SN_{S_2} cross each other. The number c_2 is given by $c_2 = 2(\Phi(k)-k'^2)/(k^2-k'^2)$, where k satisfies $(k^2-k'^2)R_3(k) = 5(k'^2-\Phi(k))$, with solution $k^2 \approx 0.93$ (see §5.2(*a*)).

(vii) $D/M = \frac{3}{4}$: L_m and L_H cross each other.

(viii) $D/M = \frac{4}{5}$: The three lines L_m, SL_S and L_{S_2} cross each other. (Equating the slopes of L_m and SL_S and the slopes of L_m and L_{S_2} shows that both events happen simultaneously.)

(ix) $D/M = 1$: L_m, L_M, L_H and L_{S_3} coalesce. L_H exists in region VIII+ and is replaced by L_{S_2} in region IX+ as the second line from which MW branches off.

If $D = 0$, $M = 0$, $D/M = \frac{1}{2}$, $D/M = \frac{3}{5}$ or $D/M = 1$, our normal form degenerates into a bifurcation of co-dimension three, and higher-order terms must be taken into account. The values c_1, c_2, $\frac{3}{4}$, $\frac{4}{5}$ for D/M give rise to events rather than to true degeneracies.

In figure 8 the stability diagrams of figure 7 are displayed in terms of bifurcation diagrams. For the regions I+ to IX+ and I− we have chosen two straight paths (*a*), (*b*) through the (μ, ν)-plane that both start in the region $\{\mu < 0, \nu < 0\}$. The path (*a*) hits first L_0 at some $\nu < 0$ and then all bifurcation lines in $\{\mu > 0\}$. The path (*b*) is obtained from path (*a*) by a parallel displacement such that the intersection with L_0 is at some $\nu > 0$. For the stability diagrams II− to IX− we have chosen one straight path that starts in $\{\mu < 0, \nu < 0\}$ and hits first H_0 and then all other bifurcation lines, i.e. the intersection with L_0 occurs at some $\nu > 0$. A parallel displacement of this path results in a trivial bifurcation diagram, i.e. only the steady-state bifurcation survives.

8. Discussion and conclusion

We have described in some detail the bifurcation behaviour near a singularity of co-dimension two of the Takens–Bogdanov type in the presence of O(2)-symmetry. Our analysis is confined to the generic case in which the essential bifurcation behaviour is determined, after scaling, by the third-order terms, and is valid when a variety of non-degeneracy conditions on the coefficients of the three nonlinear terms are satisfied. Our analysis of the problem is essentially complete. We are able to identify the three primary branches of solutions that bifurcate from the trivial state, i.e. the steady state, and the travelling and standing waves that bifurcate simultaneously. We describe their stability properties and identify a number of secondary bifurcations, of which some describe the termination on the SS-branch of the TW-branch (a pitchfork bifurcation) and the SW-branch (a saddle-loop bifurcation when $A > 0$, or a Hopf bifurcation when $A < 0$), whereas the remaining ones result in the appearance of a new secondary branch of solutions, the modulated waves (MW), that connects the SW-branches

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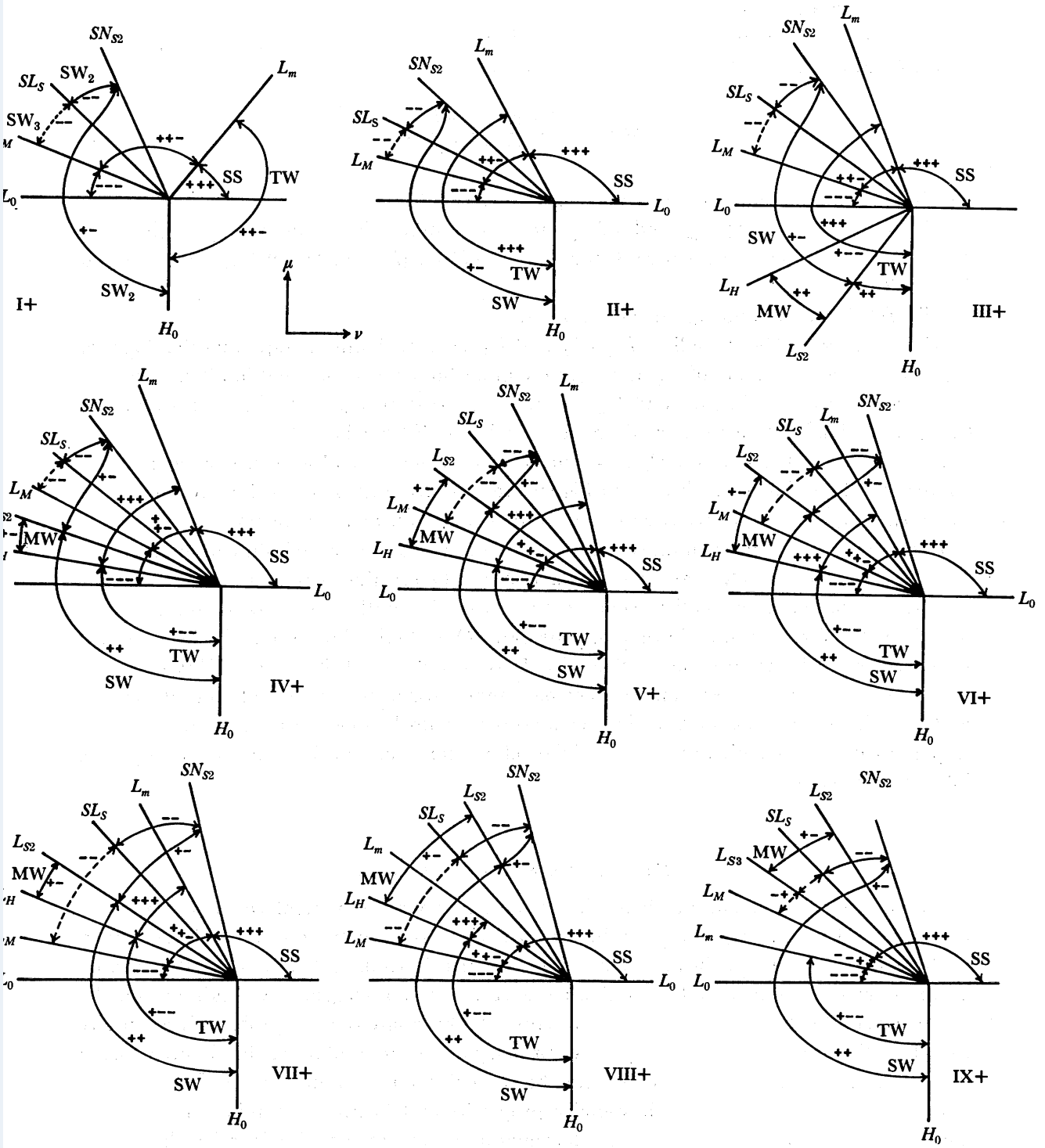

FIGURE 7. For description see opposite.

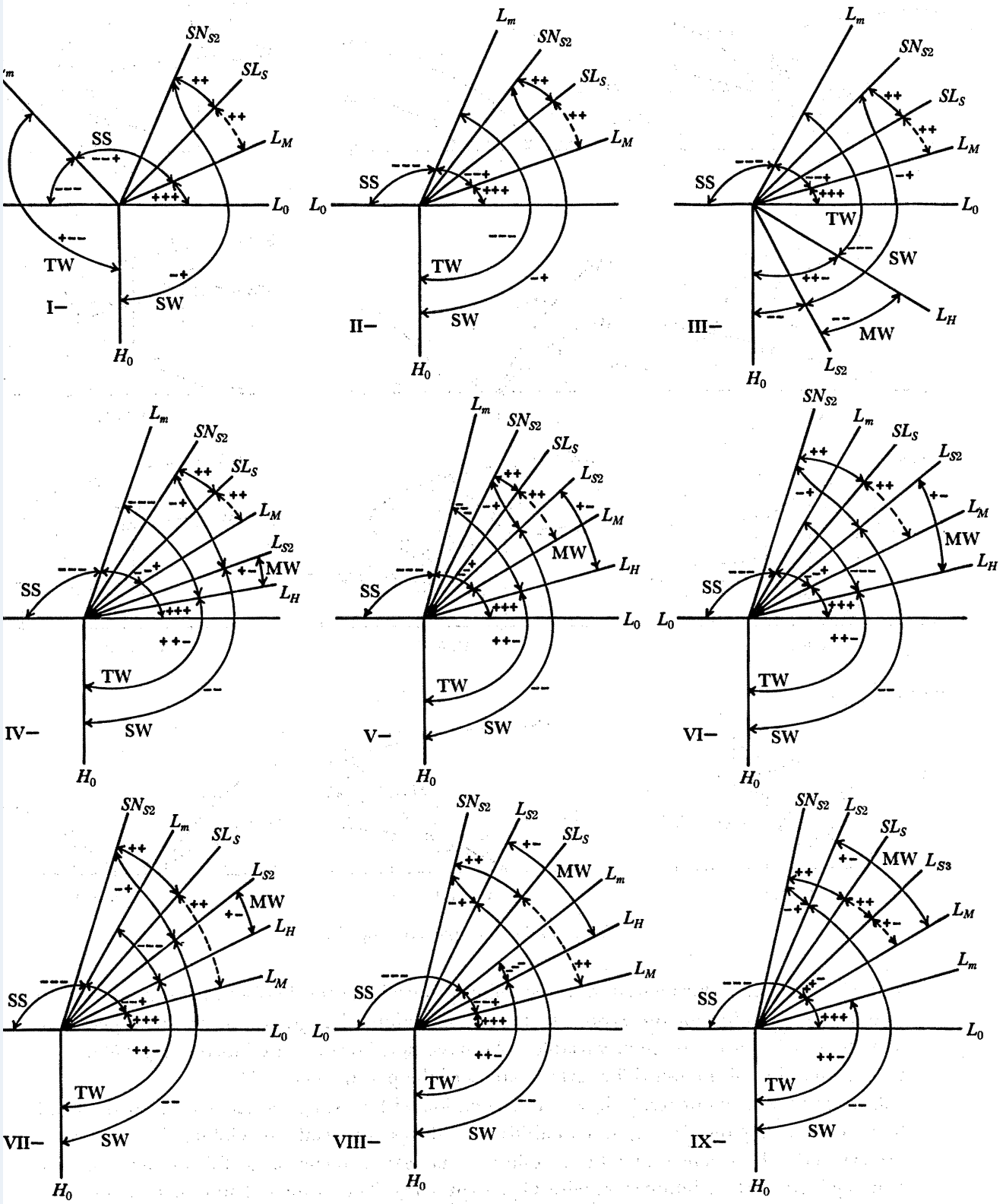

FIGURE 7. Stability diagrams corresponding to the regions marked in figure 6.

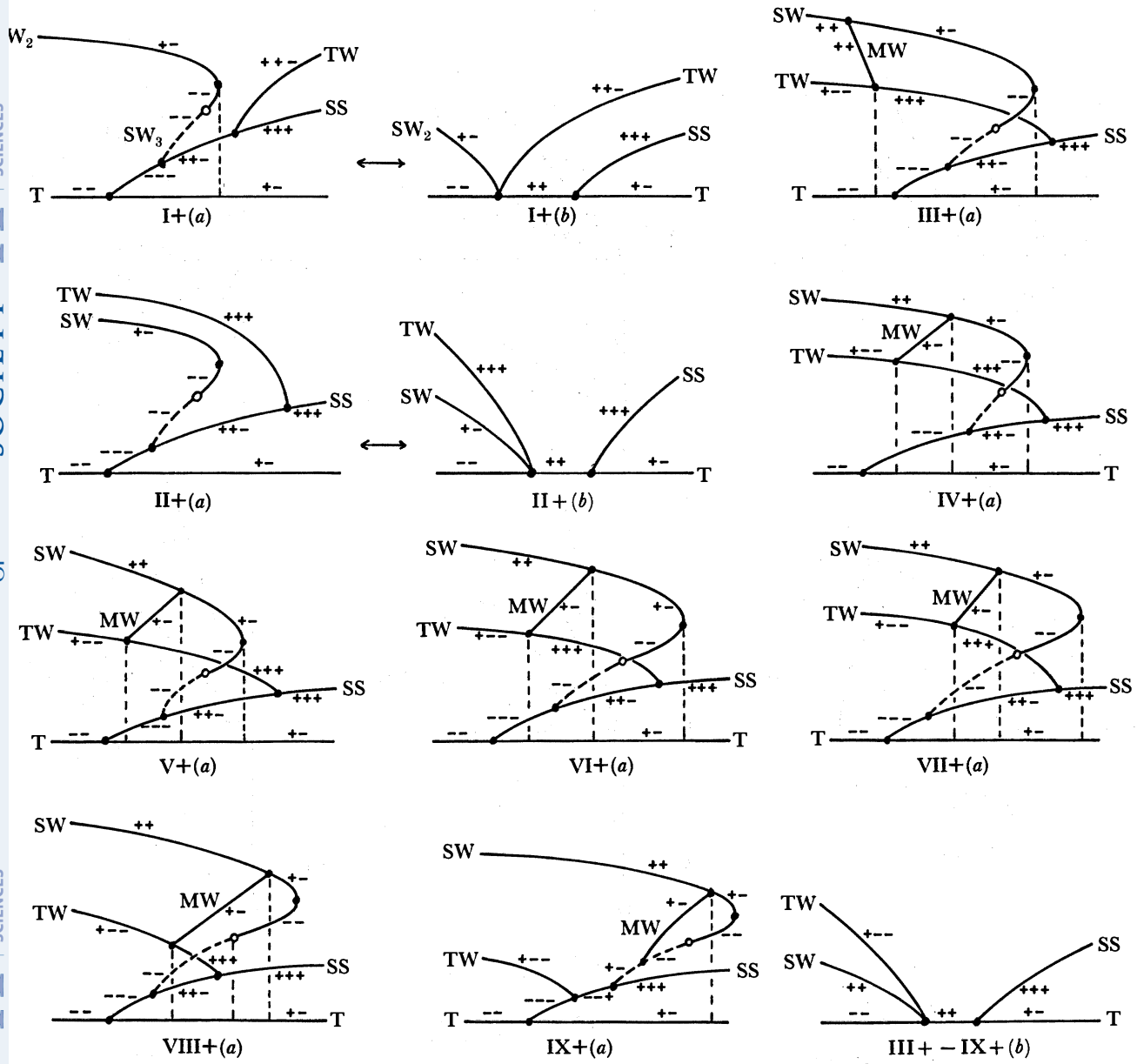

FIGURE 8. For description see opposite.

and TW-branches. Our analysis of the stability properties of the MW-branch did not lead to a sufficient condition for the existence of a tertiary bifurcation giving rise to the appearance of a three-torus in the original variables. We have not, however, excluded the possibility of an even number of three-tori for certain values of the parameters D, M.

Certain aspects of our results have a close relation with existing results. Thus near the Hopf bifurcation along the line H_0 our analysis reproduces the results of Golubitsky & Stewart (1985) and others concerning the stability of the SW-branches and TW-branches near a non-degenerate Hopf bifurcation with O(2)-symmetry. The branch of standing waves is the same as in the analogous problem with Z(2)-symmetry that has already been observed by

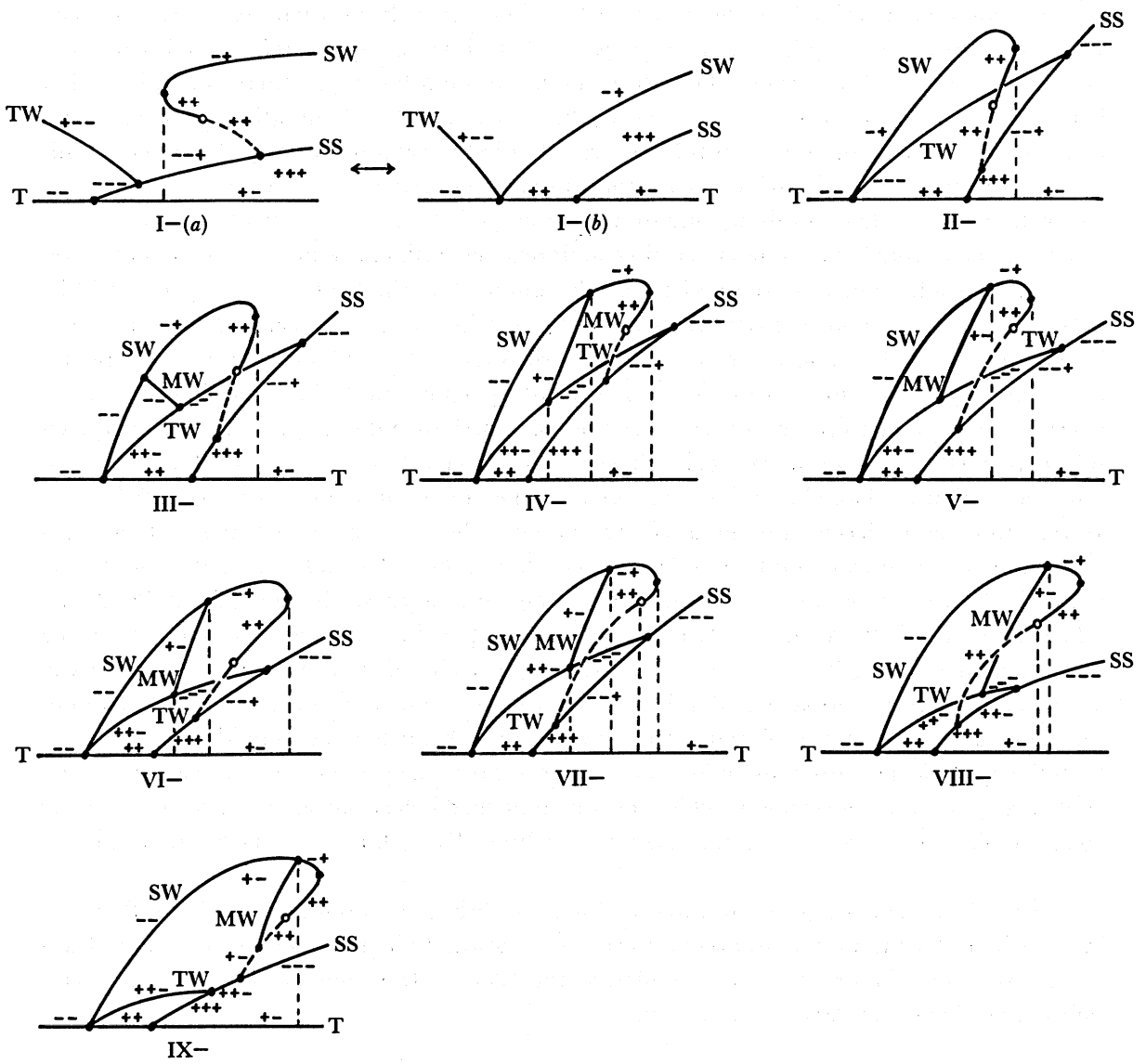

FIGURE 8. Bifurcation diagrams corresponding to the regions marked in figure 6.

Guckenheimer (1986), although in the O(2) problem the presence of the additional 'angular momentum' eigenvalue changes its stability properties. This is, of course, because the standing waves respect the Z(2)-symmetry. We also note the well-known results that the branch of steady states remains unchanged in all essential aspects when the O(2)-symmetry is added. This is in contrast with the Hopf bifurcation where the symmetry introduces a second solution branch (the travelling waves).

Throughout the paper we have made full use of a natural formulation of the problem as a classical central force problem, which has enabled us to give a simple and intuitive characterization of the various solution branches in terms of the possible motions in simple potentials. We hope that the reader will find this identification helpful in understanding the variety of bifurcation phenomena summarized in the bifurcation diagrams of §7.

The symmetry described by the group O(2) arises naturally in a variety of systems. The symmetry may appear explicitly as in physical systems with circular and reflectional symmetry, or it may be induced by periodic boundary conditions on a line when there is no distinction between the two directions (Knobloch 1986). Consequently, bifurcation problems with O(2)-symmetry play an important role in a variety of physical applications. We conclude this paper with a brief discussion of one such application. The system we consider is double-diffusive convection. Here a layer of fluid containing an imposed stabilizing gradient of a solute such as salt is heated from below. Under certain conditions, the initial instability from the no-motion (conduction) solution as the thermal Rayleigh number (i.e. the heating) is increased will be a Hopf bifurcation, and will precede a direct bifurcation to steady overturning convection. When the lateral boundaries are of the no horizontal flux type, the resulting system has Z(2)-symmetry, and the transition from oscillatory convection (i.e. standing waves) to overturning convection (i.e. steady state) can be elucidated by unfolding the Takens–Bogdanov bifurcation with Z(2)-symmetry, i.e. by choosing a second parameter (here the solutal Rayleigh number) so that the Hopf bifurcation just precedes the direct bifurcation (Knobloch & Proctor 1981). However, with *periodic* boundary conditions, travelling waves rather than standing waves are found in numerical simulations (Knobloch *et al.* 1986). To understand the interaction between the travelling waves and the steady states one must study the Takens–Bogdanov bifurcation with O(2)-symmetry. Our analysis shows that the TW-branch terminates on the SS-branch in a pitchfork bifurcation. Thus the phase velocity of the wave tends to zero linearly as the bifurcation is approached, and the wave slows down and comes to rest at the bifurcation, turning into a steady-state solution. It is expected that this behaviour should be observed in both the numerical simulations and the experiments. From the present analysis we also predict explicitly the parameter regions in which more complicated behaviour such as modulated waves could be observed. These issues are pursued elsewhere (Dangelmayr & Knobloch 1986).

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Appendix A. Formal derivation of the averaged equations

In this Appendix we give a formal derivation of the averaged equations (4.3) with (4.4) as leading terms of a perturbation analysis. The first step is to introduce a change of coordinates $(r, r', L) \to (\Theta, I, L)$ with action-angle variables (I, Θ),

$$I = \frac{1}{2\pi} \oint r'(r, E, L^2)\, \mathrm{d}r, \tag{A 1a}$$

$$\Theta = \frac{\partial S(r, I, L^2)}{\partial I}, \quad S(r, I, L^2) = \int_{r_0}^{r} r'(\tilde{r}, E(I, L^2), L^2)\, \mathrm{d}\tilde{r}. \tag{A 1b}$$

Here, $r'(r, E, L^2)$ is given by (4.2), and the integrations are carried out along a closed trajectory of the unperturbed system, $\epsilon = 0$, and r_0 is a fixed reference coordinate on it. Of course, the relation $E(I, L^2)$ occurring in S is obtained by solving $I = I(E, L^2)$ for E. In the new variables (Θ, I, L) we get the following system

$$\Theta' = W(I, L^2) + O(\epsilon), \quad W = \partial K / \partial I, \tag{A 2a}$$

$$I' = \epsilon F(\Theta, I, L^2) + O(\epsilon^2), \tag{A 2b}$$

$$L' = \epsilon L G(\Theta, I, L^2) + O(\epsilon^2), \tag{A 2c}$$

where $K(I, L^2)$ is the new hamiltonian of the unperturbed system that depends only on the action variables, and

$$\pi F = \int_{s_-}^{s_+} \frac{\mathrm{d}s}{\sqrt{[P(s)]}} \left\{ (Mr^2 + \nu)\, r'(r, E, L^2) + \left(\frac{1}{r^2} - \frac{1}{s} \right) (Dr^2 + \nu)\, L^2 \right\} \mathrm{d}s, \tag{A 3a}$$

$$G = Dr^2 + \nu, \tag{A 3b}$$

with $E = E(I, L^2)$ and $r = r(\Theta, E(I, L^2), L^2)$ being inserted according to the transformation (A 1). Expanding the functions F, G in Fourier series,

$$(F, G) = \sum_{n=-\infty}^{\infty} (F_n(I, L^2), G_n(I, L^2))\, \mathrm{e}^{\mathrm{i}n\Theta}, \tag{A 4}$$

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carrying out in (A 2) the transformation

$$(I, L) = (\tilde{I}, \tilde{L}) - \frac{i\epsilon}{W} \sum_{n \neq 0} \frac{1}{n} (G_n(\tilde{I}, \tilde{L}^2), F_n(\tilde{I}, \tilde{L}^2))\, e^{in\Theta}, \tag{A 5}$$

and dropping the tilde, we obtain

$$\Theta' = W(I, L^2) + O(\epsilon), \tag{A 6a}$$

$$I' = \epsilon F_0(I, L^2) + O(\epsilon^2), \tag{A 6b}$$

$$L' = \epsilon L G_0(I, L^2) + O(\epsilon^2), \tag{A 6c}$$

i.e. a decoupled system for (I, L). It is easy to rewrite the integration over Θ involved in the definitions of F_0, G_0 as an integration over r or $s = r^2$, respectively. Pursuing this we obtain the following representations for F_0 and G_0:

$$\pi F_0 = J_0 f(E, L^2) - L^2(\nu + DJ_1/J_0)\, J_{-1}, \tag{A 7a}$$

$$G_0 = g(E, L^2), \tag{A 7b}$$

with f and g given by (3.4) and $E = E(I, L^2)$ being inserted through (A 1). Note that F_0 involves two integrations as seen from (A 3a). If we now formally make the transformation $(I, L) \to (E, L)$ via (A 1a), we obtain exactly the averaged equation (4.6). Note further that the second term in (A 7a) vanishes if $L' = 0$, i.e. the steady states of (A 6b, c) and those of (4.6) are in 1-1-correspondence as they should be.

Appendix B. The functions $R_j(k)$, $Q(k)$

In this Appendix we present the results of asymptotic and numerical calculations concerning the functions $R_j(k)$ $(1 \leqslant j \leqslant 6)$ and $Q(k)$ that enter the persistence conditions and the variety of subordinate bifurcations. The asymptotic behaviour of these functions near $k = 0$ and $k = 1$ is summarized in table 1. Numerical plots are shown in figure 9. The latter were obtained by using the graphical capabilities of the computer algebra system SMP. The monotonicity statements given in the text rely upon these numerical computations and the asymptotics rather than upon analytical proofs.

Table 1. Asymptotic behaviour of $R_j(k)$ $(1 \leqslant j \leqslant 6)$ and $Q(k)$

	$R_1(k)$	$R_2(k)$	$R_3(k)$	$R_4(k)$
equation...	(5.5)	(5.9)	(5.14)	(5.16)
$k \to 0+$	$\frac{5}{4}k^2 + O(k^4)$	$\frac{1}{2} - \frac{1}{8}k^2 + O(k^4)$	$\frac{5}{4}k^2 + O(k^4)$	$\frac{1}{2} + \frac{1}{8}k^2 + O(k^4)$
$k \to 1-$	$\frac{1}{2} - \frac{3}{4}(k'^4/\Phi) + O(k'^4)$	$\frac{1}{5}(1 + \Phi) + O(\Phi^2)$	$-2 + 3(k'^2/\Phi) + O(k'^2)$	$\frac{2}{5}\Phi^{-1} + O(1)$

	$R_5(k)$	$R_6(k)$	$Q(k)$
equation...	(5.22)	(5.25)	(6.4)
$k \to 0+$	$\frac{5}{2} - \frac{5}{64}k^4 + O(k^6)$	$1 + \frac{1}{32}k^4 + O(k^6)$	$\frac{2}{15} - \frac{1}{320}k^4 + O(k^6)$
$k \to 1-$	$2 + 2(k'^2/\Phi) + O(k'^2)$	$\frac{2}{5}\Phi^{-1} + O(1)$	$\frac{1}{2}\Phi + O(\Phi^2)$

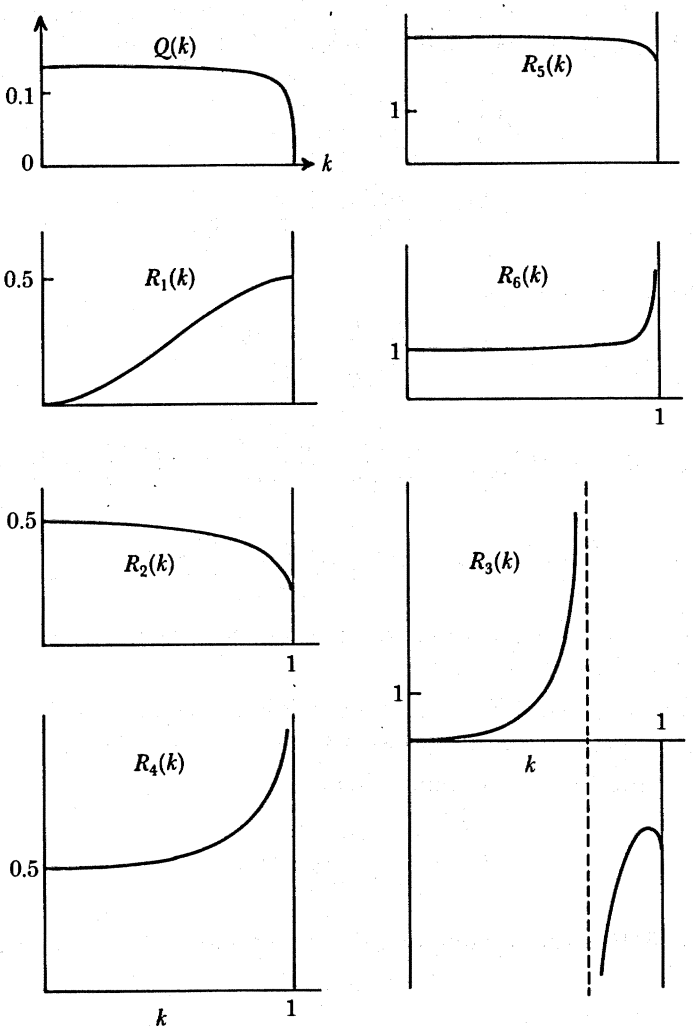

FIGURE 9. Plots of the functions $R_j(k)$ $(1 \leqslant j \leqslant 6)$ and $Q(k)$.

APPENDIX C. STABILITY OF THE BRANCH OF MODULATED WAVES NEAR THE SADDLE-LOOP BIFURCATION FOR $A > 0$

The MW-branch is specified by the zeroes of the equations

$$f(E, L^2) = 0, \quad g(E, L^2) = 0, \tag{C 1}$$

where

$$E = E(k^2, \rho), \quad L^2 = L^2(k^2, \rho), \tag{C 2}$$

and k^2, ρ are the two moduli introduced in (4.7) and (4.9). To determine the stability of a solution of (C 1), we have to analyse (4.13), i.e. the eigenvalues of the 2×2 matrix

$$\Lambda = \begin{bmatrix} f_E & 2Lf_{L^2} \\ Lg_E & 2L^2 g_{L^2} \end{bmatrix}. \tag{C 3}$$

Because Λ has to be evaluated along the solution of (C 1) that is given parametrically in terms of k and $\rho(k)$ by (6.2)–(6.4), we express the entries of (C 3) also in terms of (k,ρ). We note that

$$\left.\begin{aligned} d\frac{\partial k^2}{\partial E} &= \frac{\partial L^2}{\partial \rho}, \quad d\frac{\partial k^2}{\partial L^2} = -\frac{\partial E}{\partial \rho}, \\ d\frac{\partial \rho}{\partial E} &= -\frac{\partial L^2}{\partial k^2}, \quad d\frac{\partial \rho}{\partial L^2} = \frac{\partial E}{\partial k^2}, \end{aligned}\right\} \tag{C 4}$$

where d is the determinant

$$d = \frac{\partial E}{\partial k^2}\frac{\partial L^2}{\partial \rho} - \frac{\partial E}{\partial \rho}\frac{\partial L^2}{\partial k^2}. \tag{C 5}$$

Thus,

$$d\,\mathrm{tr}\,\Lambda = d(f_E + 2L^2 g_{L^2}) = \frac{\partial f}{\partial k^2}\frac{\partial L^2}{\partial \rho} - \frac{\partial f}{\partial \rho}\frac{\partial L^2}{\partial k^2} + 2L^2\left(\frac{\partial g}{\partial \rho}\frac{\partial E}{\partial k^2} - \frac{\partial E}{\partial \rho}\frac{\partial g}{\partial k^2}\right), \tag{C 6}$$

$$d\det\Lambda = 2L^2 d(f_E g_{L^2} - g_E f_{L^2}) = 2L^2\left(\frac{\partial f}{\partial k^2}\frac{\partial g}{\partial \rho} - \frac{\partial f}{\partial \rho}\frac{\partial g}{\partial k^2}\right). \tag{C 7}$$

The partial derivatives (C 4) are computed by using the representation (4.10), whereas the derivatives of f, g follow from (4.4)–(4.7).

We wish to calculate (C 6) and (C 7) near the saddle-loop end of the MW-branch for $A > 0$. To do this, we use the persistence conditions (6.2a), (6.3) and (6.4), and consider the limit $k \to 1-$. We first compute the determinant (C 5)

$$d = \frac{4\mu^5 \bar{\rho}^4 k'^2}{A^3(3+2\bar{\rho})^6} + O(k'^4), \tag{C 8}$$

where $\bar{\rho} = 5D/(M-5D)$ is the ρ-value of MW at the saddle loop ((6.2a) and (6.9)). Next, we obtain for $k \to 1-$,

$$\frac{\partial g}{\partial \rho} = -\frac{2\mu D(1-3\Phi)}{A(3+2\bar{\rho})^2} + O(\Phi^2), \tag{C 9a}$$

$$\frac{\partial g}{\partial k^2} = \frac{2\mu D\bar{\rho}}{A(3+2\bar{\rho})^2}\{(3+2\bar{\rho})\Phi' + \bar{\rho} + 2\} + O(\Phi), \tag{C 9b}$$

$$\frac{\partial f}{\partial k^2} = -\frac{2\mu^3 D\bar{\rho}}{A^2(3+2\bar{\rho})^3}\{2(1+\bar{\rho})\,\Phi' + \bar{\rho}(1+2\bar{\rho})\,\Phi\Phi'\} + O(\Phi^2\Phi'), \tag{C 9c}$$

$$\frac{\partial f}{\partial \rho} = -\frac{2\mu^3 D}{A^2(3+2\bar{\rho})^4}\{2(1+\bar{\rho}) - \bar{\rho}(1+2\bar{\rho})\,\Phi\} + O(\Phi^2), \tag{C 9d}$$

where $\Phi' = \mathrm{d}\Phi/\mathrm{d}k^2$. Because $\Phi' \to -\infty$ as $k \to 1-$, we find that, at leading order,

$$\frac{\partial f}{\partial k^2}\frac{\partial g}{\partial \rho} - \frac{\partial f}{\partial \rho}\frac{\partial g}{\partial k^2} = \frac{8D^2\bar{\rho}^2\mu^4}{A^3(3+2\bar{\rho})^4}\Phi\Phi' + O(\Phi^2\Phi'),$$

and hence

$$\det\Lambda = -\frac{16\mu^2}{3+2\bar{\rho}}\left(\frac{D}{A}\right)^2\left(\frac{1+\bar{\rho}}{\bar{\rho}}\right)^2\frac{\Phi\Phi'}{k'^2} + O(\Phi^2\Phi'/k'^2). \tag{C 10}$$

Because $\det A > 0$, there are either two stable or two unstable eigenvalues near the saddle loop. To determine which case arises we must compute the trace. This is a delicate calculation. The leading terms are those proportional to Φ'. Thus

$$d \operatorname{tr} A = \frac{16\mu^5\rho^2(1+\rho)\,\Phi'}{A^4(3+2\rho)^7}\left\{-\frac{4M}{15}\mu\rho^2+\tfrac{1}{3}\nu A(3+2\rho)^2-2\mu D(1+\rho)\right\}+\ldots. \tag{C 11}$$

We next evaluate the expression (6.2a) in the limit $k\to 1-$

$$\rho = \frac{5D}{M-5D}\left\{1-\frac{3M-5D}{M-5D}\tfrac{1}{2}\Phi+\ldots-\frac{1}{2}\frac{k'^2}{\Phi}\frac{M}{M-5D}+\ldots\right\}, \tag{C 12}$$

and similarly for the persistence condition (6.3)

$$A\nu = \frac{2D\mu}{3M-5D}\{M-\tfrac{15}{2}D\Phi+\ldots\}. \tag{C 13}$$

Here the first ... in (C 12) represents all powers of Φ, which are all larger than the term k'^2/Φ as $k\to 1-$. No term k'^2/Φ occurs in (C 13). It is very easy to show that the trace vanishes identically if only the terms of order $\Phi'\Phi^n$, $n = 0, 1, 2, \ldots$ are kept. Hence the leading term involving Φ' in the right-hand side of (C 11) is of order $\Phi' k'^2/\Phi$, which tends to zero as $k\to 1-$. Consequently we need the constant term in the asymptotic expansion of $d \operatorname{tr} A$. To obtain this it is more convenient to work with the representations (4.6) and (4.4b). With $g = 0$, we obtain

$$df_E = d\left\{-\frac{4M\mu}{15A}+\tfrac{4}{15}(3M-5D)\frac{J_1}{J_0}\right\}$$
$$+\left\{-\tfrac{1}{3}D\mu\frac{J_1}{J_0}+\tfrac{4}{5}ME-\frac{4M}{15A}\mu^2\right\}\left\{\frac{\partial L^2}{\partial\rho}\frac{\partial}{\partial k^2}\left(\frac{J_2}{J_0}\right)-\frac{\partial L^2}{\partial k^2}\frac{\partial}{\partial\rho}\left(\frac{J_2}{J_0}\right)\right\}. \tag{C 14}$$

It is easy to see that the desired constant term in df_E that contributes to $d \operatorname{tr} A$ is given by

$$\frac{4M}{15}\left(3E-\frac{\mu^2}{A}\right)\left(\frac{\partial L^2}{\partial\rho}\frac{\partial s_1}{\partial k^2}-\frac{\partial L^2}{\partial k^2}\frac{\partial s_1}{\partial\rho}\right) = -\frac{32M\mu^6\bar{\rho}^4(1+\bar{\rho})}{15A^4(3+2\bar{\rho})^7}+\ldots. \tag{C 15}$$

Similarly we get

$$dg_{L^2} = D\left\{\frac{\partial E}{\partial k^2}\frac{\partial}{\partial\rho}\left(\frac{J_1}{J_0}\right)-\frac{\partial E}{\partial\rho}\frac{\partial}{\partial k^2}\left(\frac{J_1}{J_0}\right)\right\}, \tag{C 16}$$

and the relevant term of $2L^2 g_{L^2}$ contributing to $d \operatorname{tr} A$ has the form

$$2L^2D\left(\frac{\partial E}{\partial k^2}\frac{\partial s_1}{\partial\rho}-\frac{\partial E}{\partial\rho}\frac{\partial s_1}{\partial k^2}\right) = -\frac{16D\mu^6\bar{\rho}^2(1+\bar{\rho})^2}{A^4(3+2\bar{\rho})^7}+\ldots. \tag{C 17}$$

Adding the right-hand side of (C 15) and (C 17) and dividing the result by d yields for the leading term of the trace

$$\operatorname{tr} A = -\tfrac{4}{15}M\mu\left(\frac{1+\bar{\rho}}{\bar{\rho}}\right)k'^{-2}+O(\Phi/k'^2). \tag{C 18}$$

Because $\mu < 0$, we conclude that $\operatorname{sgn}\operatorname{tr} A = \operatorname{sgn} M$; hence the stability assignment of the MW-branch near the saddle-loop end is ($\operatorname{sgn} M$, $\operatorname{sgn} M$). Moreover, the asymptotic behaviour of $\operatorname{tr} A$ and $\det A$ implies that both eigenvalues are real and one is of order $1/k'^2$, whereas the other is of order Φ^3/k'^2.